

ELEMENTS
OF
SOLID GEOMETRY,
Algebraically Demonstrated.
In THREE BOOKS.
WITH AN
APPENDIX.

By HENRY GORE, Teacher of the
MATHEMATICKS in *Manchester*.



L O N D O N:

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TO THE
RIGHT HONOURABLE
THE
EARL of *DALKEITH*.

MY LORD,

AT the same Time YOUR LORDSHIP does me the highest Honour in taking this little book under Your Protection, You discover that Goodness and Compassion, which, with the Progress YOUR LORDSHIP has so early made in your Studies,

DEDICATION.

dies, justly give the World the greatest Expectations.

YOUR LORDSHIP must receive frequent Addresses of this Sort from Men of the First Learning and Genius ; be mine the Merit of shewing Them their PATRON, and of being for ever, with the utmost Gratitude, MY LORD,

YOUR LORDSHIP'S

Most Obedient, and

Most Devoted Humble Servant,

Henry Gore.

TO THE

READER.

THIS Treatise consisteth of Three Books and an Appendix: The First Book contains all those Propositions in the Eleventh and Twelfth Books of EUCLID's Elements of Geometry, which concern Analogy or Proportion; all the other Propositions ought rather to be illustrated by a proper Position of Lines and Planes, than demonstrated by any Method whatsoever. These Books are of such known Worth, that it is needless to say any thing in their Commendation here.

The Second Book contains the Thirteenth, Fourteenth, and Fifteenth Books of EUCLID's Elements; with vast Improvements in the Properties of the Regular or Platonic Bodies: Shewing great Variety of Proportions, which they bear to one another, and also to their Circumscribing and Inscribed Spheres, and to the Sphere which passeth through the Middle of each Side of each of them. The Worth of these Three Books is evident from what PROCLUS relates, viz. That EUCLID composed his whole Fifteen Books of Geometry for the sake of these Three.

The Third Book contains my Select Theorems out of ARCHIMEDES's Treatise of the Sphere and Cylinder, with several others (no less curious) on the same Subject; invented by the learned ANDREW TACQUET. That ARCHIMIDES looked upon these Theorems as
a the

TO the READER.

the most valuable of all his wonderful Inventions, seems evident by his chusing the Sphere, Cone and Cylinder, inscrib'd one in another for his Monument.

The Reasons which induced me to make use of an Algebraical Demonstration were many, one of which is the following :

I observed that several of the Demonstrations according to the Method of the Ancients are fetch'd, by intermediate Conclusions, from Principles so very remote, and require so long a Series of Mediums to be employ'd about them, that not only careless and indolent, but even sedulous and heedful Readers, though endued with invincible Patience, do find themselves many times hard put to it, to carry along such a Chain of Inferences in their Minds, as clearly to discern whether the whole Ratiocination be coherent, and all the Particulars have their due Strength and Connection. But in the Algebraical Way, especially where the Process is carried on by Steps, and a proper Margent kept, the References are few, the Chain short, and the whole Connection plain. In short, by this Method we arrive at Truth by an easy, cheap, and compendious Way; and it is for that very Reason, by so much more perfect than any other Method, by how much fewer, and those more facil, Steps it takes to obtain its Ends.

I know very well that this Method of demonstrating Geometry by Algebra hath both Advocates and Adversaries; I shall therefore leave the Point to be determin'd by them that are in love with Contention.

Manchester,
Feb. 25. 1732-3.

FAREWELL.

N. B. For

To the READER.

N. B. For the sake of those who have a Mind to reduce the following Theorems to a Calculus, I have given the following Expressions in Numbers

$$S=5877852. \quad s=,8090170 \quad \sqrt{\frac{4S^2-b}{6}} = \sqrt{\frac{4S^2}{6}}$$

$$1=1.308924 \quad \sqrt{4S^2-6}. \sqrt{12S^2-36}=1,617991.$$

If I had this Work to begin again, I would substitute Letters for these Numbers, and then there would not be a long or troublesome Expression in the whole Book.



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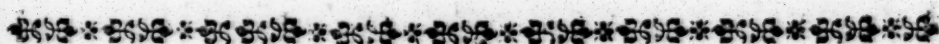
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ELEMENTS

O F

Solid GEOMETRY, &c.



BOOK I.

*Containing most of the Eleventh and Twelfth
Books of EUCLID'S Elements of
GEOMETRY.*

DEFINITIONS.

1.  Solid is that which hath Length,
Breadth, and Thickness.
2. The Bounds of a Solid are
Superficies.
3. Like solid Figures, are such
as are contained under like Plains, equal in
Number.
4. Equal and like solid Figures, are such as are
contained under like Plains, equal, both in Multi-
tude and Magnitude.

B



5. A

- Fig. 1. 5. A Parallelepipedon is a solid Figure, contained under six quadrilateral Figures, whereof those that are opposite are equal, alike, and alike situated.
- Fig. 2. 6. A *Prism* is a solid Figure, contained under five Plains at least, the two opposite of which are alike, equal, and parallel, and alike situated.
7. A *Prism* hath an additional Name, derived from the Figure of its Base; for if the Base be triangular, it is called a *Triangular Prism*. If the Base be a *Pentagon*, it is called a *Pentagonal Prism*, &c.
- Fig. 3. 8. A *Cylinder* is a round Solid, like the rolling Stone of a bowling Green, whose two Ends are equal, and parallel Circles.
- Fig. 5. 9. A *right Pyramid*, is a solid Figure, whose Base may be any *regular Polygon*, but its Sides equal, plain *Isosceles Triangles*, whose Number is equal to the Number of the Sides of the Polygonous Base, and whose several Tops meet in a Point perpendicular to the Center of its Base, called the *Vertex*.
10. A *Pyramid* hath also an additional Name, derived from the Figure of its Base; as a *Triangular, Pentagonal, Pyramid*, &c.
- Fig. 6. 11. A *Sphere* or *Globe*, is a solid Figure exactly round every way, having all the Parts of its Surface equally distant from a point within, called the *Center of the Sphere*.
12. The *Axis* or *Diameter* of a Sphere, is a right Line passing through the Center of the Sphere, and terminated in two opposite Points in the Superficies, as, *a b*.
- Fig. 4. 13. A *Cone* is a round Solid, in Form like a Sugar-loaf, whose Base is the circular End, and its Axis the right Line reaching from the Center of the Base, unto the Point at top, called the *Vertex*.
14. Like

14. Like *Cones* and *Cylinders*, are they whose Axes and Diameters of their Bases are proportional.

15. Solids are said to be reciprocally proportional, when the Proportion both begins and ends in the same Solid.

16. A solid Figure is said to be inscribed in a solid Figure, when all the Angles of the inscrib'd Figure are comprehended, either within the Angles, or in the Sides, or in the Plains, of the Figure wherein it is inscribed.

17. Also a solid Figure is then said to be circumscribed about a solid Figure, when either the Angles, or Sides, or Plains of the circumscribed Figure, touch all the Angles of the Figure which it contains.

In order to a thorough understanding of the following Demonstrations, it will be necessary to consider the following Solids, to be generated as followeth.

18. All *Parallelepipedons* to be compos'd of an infinite Number of infinitely thin contiguous square or oblong Plains, similar, equal, and alike situated, and placed at Right Angles to a Plain passing through their Middles or Centers.

19. All *Prisms* to be composed of an infinite Number of infinitely thin polygonous Plains, like and equal to one another, and placed as in the last.

20. All *Cylinders* to be composed of an infinite Number of infinitely thin circular Plains, whose Diameters are all equal, and placed as before.

21. All right *Pyramids* to be composed of an infinite Number of infinitely thin polygonous Plains,

the Length of whose Sides still diminish 'till they end in a Point at the *Vertex*.

22. All right *Cones* to be composed of an infinite Number of infinitely thin circular Plains, whose Diameters still decrease 'till they end in a Point at the *Vertex*.

23. *Corol. 1.* From what hath been said it is evident, that all *Parallelepipedons*, *Prisms*, *Pyramids*, *Cylinders*, and *Cones*, that are composed of an equal Number, of equal Plains respectively, are respectively equal.

24. *Corol. 2.* It plainly followeth from *Art. 18, 19, 20.* that the Solidities of all *Parallelepipedons*, *Prisms*, and *Cylinders*, are found by multiplying the Area of the Base, *i. e.* Area of one of the composing Plains, into the Altitude, *i. e.* into the Number of all the Plains whereof they are composed.

For in these Solids, the Plains that compose any of them are equal, but the Sum of any Number of equal Terms, *i. e.* one of the Terms so often repeated as are the Number of Terms, is equal to the Product of one of the Terms, multiplied into the Number of all the Terms, but their Solidities are equal to the Sum of all those Plains, wherefore, &c.

25. *N. B.* I shall all along represent the Altitude of the foregoing Solids, *i. e.* the Number of Plains which compose them by, *A* or *a*.

26. Every Sphere to be composed of an infinite Number of Cones, whose Bases are infinitely little, and taken all together compose the Surface of the Sphere, and whose Vertexes meet all together in the Center of the Sphere, just as a Circle may be supposed to be composed of an infinite Number of Isosceles

Isoceles Triangles, the aggregate of whose Bases makes the Circumference, and their common *Vertex* is at the Center.

I shall all along suppose the Reader so well acquainted with plain Geometry, as to know that the

27. Superficies of any Square, or Parallelogram, is found by multiplying the Length into the Breadth; or by multiplying one Side into another next adjacent Side.

28. That the Area of a Triangle is found by multiplying half the Perpendicular into the whole Base; or half the Base into the whole Perpendicular.

29. That the Area of any regular Poligon is found by multiplying half the Sum of the Sides, into a Perpendicular from the Center to the middle of a Side.

30. That there is a constant *Ratio* between the Diameter of a Circle, and its Circumference, which *Ratio* we will denote by the Letter (*e.*) so that the Diameter of a Circle being multiplied by the *Ratio e*, the Product will be the Circumference; and the Circumference divided by the said *Ratio*, the Quotient will be the Diameter.

31. That the Area of a Circle is found, by multiplying half the Circumference by the Radius, or Semidiameter; or by multiplying a fourth Part of the Circumference into the whole Diameter; or a fourth Part of the Diameter into the Circumference: or (which is the same thing) by multiplying the Square of the Radius into the common *Ratio*.

32. These things being premis'd, let *L*, and *B*, be put for the two contiguous Sides of a Square, or
Paral-

Parallelogram; l , and b , for the two like Sides of any other Square, or Parallelogram, which may be greater, or less, than the former. Then will LB , and lb , be the Expressions for the Areas of those Squares, or Parallelograms. *per Art. 27.*

33. Also let N , and n , be put for the Number of Sides of any regular *Poligon*; B , and b , for the Length of the Sides, and P , and p , for the Perpendiculars from their Center to the Middle of each Side, then will $\frac{1}{2}NB P$, and $\frac{1}{2}nbp$, be the Expressions for the Areas of those *Poligons*. *per Art. 29.*

34. If the Diameter of a Circle be $2R$, and the Diameter of another $2r$, their Circumferences will be $2Re$, and $2re$, *per Art. (30)* and their Areas will be R^2e , and r^2e , *per Art. 31.*

From what hath been laid down, the following Theorems are self-evident.

THEOREM I.

35. The Areas or superficial Content of all Parallelepipedons, Prisms, and Cylinders, are equal to the Area of all their circumscribing Plains, *i. e.* to the Circuit of their Base, multiplied into their Height. *Example. 1.* The Area of all Parallelepipedons, (in our way of Notation) is equal to $2L + 2B \times A = \overline{L+B} \times 2A$. *i. e.* If you multiply the Sum of two contiguous Sides of a Parallelepipedon, by twice the Length, the Product will be the Area without the Ends; and the whole Area is expressed by $\overline{L+B} \times 2A + 2BL$. *i. e.* twice the Product of two contiguous Sides, added to the Product of twice the Altitude multiplied into

into the Sum of the said Sides, that Sum will be the whole Area.

COROLLARIES.

36. If it be a square Parallelepipedon, *i. e.* If $L=B$, the Expression without the Ends is $4AB$, and for the whole Area it is $4AB + 2B^2 = 2B \times 2\overline{A+B}$.

37. If the Altitude be also equal to the Side of the Base, *i. e.* if $L=B=A$, *i. e.* if it be a Cube, the Expression for the Area will be $6B^2$.

38. *Example 2.* The Superficies of all Prisms without the Ends, is expressed by NAB , *i. e.* multiply the Length of a Side, the Number of Sides, and the Length of the Prism, continually one into another; and that Product will be the Area.

39. The Expression for the whole Area is $NBA + NBP = NB \times \overline{A+P}$, *i. e.* multiply the Side, into the Number of Sides, and that Product into the Sum of the Length of the Prism, and Perpendicular from the Center of the Base to the middle of a Side; and that Product will be the Area.

COROLLARIES.

40. If $N=4$, *i. e.* if the Prism have a square Base, *i. e.* if it be a square Parallelepipedon, the Expression for the Superficies without the Ends is $4BA$, and for the whole Superficies $4AB + 2B^2 = 2B \times 2\overline{A+B}$, the very same with *Art.* 36.

41. If $N=5$, *i. e.* if it be a pentagonal Prism, the Expression will be $5BA$, without the Ends, and for the whole Superficies $5BA + 5BP = 5B \times \overline{A+P}$.

42. *Ex-*

42. *Example 3.* The Expression for the Superficies of a Cylinder without the Ends is $2 A R e$, *i. e.* multiply twice the Radius or the Diameter, and common *Ratio*, and Altitude of the Cylinder, continually one into another, and the last Product will be the Superficies.

43. And the Expression for the whole Superficies is $2 R e A + 2 R^2 e = 2 R e \times \overline{A + R}$, *i. e.* multiply the Circumference of the Base, into the Sum of the Length, and Radius; that product will be the whole Superficies.

COROLLARY.

44. If $A = 2 R$, which is the Case of a Cylinder circumscribing a Sphere, the Expression for the Superficies without the Ends is $4 R^2 e$, and for the whole Superficies it is $6 R^2 e$.

THEOREM II.

45. The Superficies of all right Pyramids, and Cones, is equal to the Rectangle of the Circuit of the Base, into half the Side, or of the whole Side, into half the Circuit of the Base.

N. B. By the Side, or flant Height of a Pyramid, or Cone, I mean the Length of a Perpendicular, drawn from the *Vertex*, to the outside of the Base.

46. *Example 1.* Suppose S to denote the Length of the Side, or flant Height of a Pyramid, or Cone; then the Expression for the Superficies of a Pyramid, without the End, or Base is, $NB \times \frac{S}{2}$, and for the whole Superficies it is $NB \times \frac{S}{2} + NB \times \frac{P}{2} = NB \times \frac{S + P}{2}$

Example

Example 2. The Expression for the Superficies of a *Cone*, without its Base is $Re \times S$, and the Expression for the whole Superficies is $SRe + Re^2 = Re \times S + R$.

T H E O R E M III.

47. If the Altitude of a *Parallelepipedon*, &c. be called A , as before, the Expression for its Solidity will be ABL , per *Art. 24. i. e.* multiply the Product of the two contiguous sides, into the Altitude, and that Product will be the Solidity.

48. *Example 2.* The Expression for the Solidity of a *Prism* is $\frac{1}{2} ANBP$, per *Art. 24. i. e.* half the last Product of the continual Multiplication of all the contiguous sides, the Altitude, and Perpendicular from the Center of the Base to the middle of a Side, will be the Solidity.

49. *Example 3.* The Expression for the Solidity of a *Cylinder* is AR^2e , per *Art. 24. i. e.* the last Product of the Square of the Radius of the Base, the constant Ratio, and Altitude, is the Solidity.

C O R O L L A R I E S.

50. If the *Parallelepipedon* have a Square Base, *i. e.* if $L=B$, then its Solidity will be expressed by AB^2 ; and if the Length or Altitude be also equal to the Side, *i. e.* if $A=B=L$, the Expression will be B^3 .

51. If $N=4$, the *Prism* is called a Square *Prism*, or indeed a Square *Parallelepipedon*, and the Expression for its Solidity will be $2ABP$. But because in this Case $P=\frac{1}{2}B$, the Expression will be

C

$2AB$

Elements of SOLID GEOMETRY

$2 AB \times \frac{B}{2} = AB^2$, the very same with the Expression for the Square *Parallelepipedon* in the last.

52. If in a *Cylinder* we have $A = 2R$, *i. e.* if the Altitude be equal to the Diameter, the Expression for its Solidity will be $2R^3e$.

A X I O M S.

54. Those Things that are equal to the same Things, are equal also amongst themselves.

55. If to Equals, you add Equals, the Sums will be equal.

56. If from Equals you subtract Equals, the Remainder will be equal.

57. If equal Quantities be multiplied by equal Quantities, the Products will be equal.

58. If equal Quantities, be divided by equal Quantities, their Quotients will be equal.

59. Parallel Lines have a common Perpendicular, *i. e.* the right Line that is perpendicular to one of them, is perpendicular to the other.

60. Two perpendicular Lines intercept equal Parts of their Parallels.

61. The Whole is equal to the Sum of all its Parts.

62. Of PROPOSITIONS, some propose some Thing to be done, and are called *Problems*; in others, we proceed no further than bare Contemplation, which therefore are called *Theorems*.

P R O P.

Algebraically Demonstrated.

I :

P R O P. I.

63. If two right Lines BCD , GHI , be cut by F parallel Plains, BG , CH , DI , they will be cut proportionally, *i. e.* $BC : CD :: GH : HI$. Draw BNI .

Then $\left\{ \begin{array}{l} 1 \mid BC : CD :: BN : NI \\ 2 \mid GH : HI :: BN : NI \end{array} \right\}$ per 2.6.
But $\left\{ \begin{array}{l} 3 \mid BC : CD :: GH : HI, \text{ per 11.5.} \end{array} \right\}$
Q. E. D.

P R O P. II.

64. *Parallelepipedons* ABL , abl , which have the same, or equal Bases, are one to another as their Altitudes or Lengths.

For $\left\{ \begin{array}{l} 1 \mid ABL = \\ 2 \mid abl = aBL (\text{because } BL = bl) \end{array} \right\} = \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \left\{ \begin{array}{l} \text{Parall.} \\ \text{p. art. 47} \end{array} \right\}$
But $\left\{ \begin{array}{l} 3 \mid ABL : aBL :: A : a, \text{ per 16.6.} \end{array} \right\}$ Q. E. D.

P R O P. III.

65. *Parallelepipedons* ABL , abl , which have the same, or equal Altitudes, are one to another as their Bases.

But $\left\{ \begin{array}{l} 1 \mid ABL = \\ 2 \mid abl = Abl (\text{because } A = a, \text{ p. hyp.}) \end{array} \right\} = \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \left\{ \begin{array}{l} \text{art.} \\ 47. \end{array} \right\}$
For $\left\{ \begin{array}{l} 3 \mid ABL : Abl :: BL : bl, \text{ per 16.6.} \end{array} \right\}$ Q. E. D.

P R O P. IV.

66. Like *Parallelepipedons* ABL , abl , are in triplicate Proportion of their homologous Sides.

For $\left\{ \begin{array}{l} 1 \mid AB : ab :: A^2 : a^2, \text{ per Hyp. and 19.6.} \\ 2 \mid L : l :: A : a, \text{ per Hyp.} \\ 3 \mid ABL : abl :: A^3 : a^3, \text{ Q. E. D.} \end{array} \right\}$
1 x 2

C 2

P R O P.

P R O P. V.

67. If *Parallelepipedons* be equal, their Bases and Altitudes are reciprocally proportional.

$$\begin{array}{l|l|l} \text{For} & 1 & ABL = a b l, \text{ per Hyp.} \\ \text{But} & 2 & LB : lb :: a : A, \text{ per 16.6. Q. E. D.} \end{array}$$

P R O P. VI.

Fig. 7. 68. A *Parallelepipedon* HD, made of three proportional Lines, *A*, *B*, *C*, is equal to the *Parallelepipedon* AC made of the Mean.

$$\begin{array}{l|l|l} \text{For} & 1 & A : B :: B : C, \text{ per Hyp.} \\ 1 \therefore & 2 & AC = B^2, \\ 2 \times B & 3 & ABC = B^3, \text{ Q. E. D.} \end{array}$$

P R O P. VII.

Fig. 10. 69. If there be four right Lines proportional, as *A*, *B*, *C*, *D*, the solid *Parallelepipedons*, *A*, *B*, *C*, *D*, being alike, and in like sort described from them, shall be proportional.

$$\begin{array}{l|l|l} \text{For} & 1 & A : B :: C : D, \text{ per Hyp.} \\ \text{And} & 2 & A^3 : B^3 :: C^3 : D^3, \text{ per cubing.} \\ \text{But} & 3 & \text{Paral. } A : \text{Paral. } B :: \text{Paral. } C : \text{Paral. } D. \\ & & \text{Q. E. D.} \end{array}$$

P R O P. VIII.

70. *Prisms* of the same Height, are one to another as their Bases.

$$\begin{array}{l|l|l} \text{For} & \left\{ \begin{array}{l} 1 \frac{1}{2} ANBP \\ 2 \frac{1}{2} anbp = \frac{1}{2} Anbp, \text{ per hyp.} \end{array} \right\} & = \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \text{ per Art. 48.} \\ \text{But} & 3 \frac{1}{2} ANBP : \frac{1}{2} Anbp :: \frac{1}{2} NBP : \frac{1}{2} nbp, \text{ per 16.6,} \end{array}$$

COROL.

C O R O L L A R I E S.

71. 1. Hence, if the *Prisms* are both of one kind, *i. e.* both Triangular, Pentagonal, &c. then is $N=n$, and they will be one to another as $\frac{1}{2} BP : \frac{1}{2} bp$, or as $BP : bp$.

72. 2. If $P=p$, they will be as $\frac{1}{2} NB : \frac{1}{2} nb$, or as $NB : nb$.

73. 3. If $B=b$, they will be as $\frac{1}{2} NP : \frac{1}{2} np$, or $NP : np$.

74. 4. If $BN=nb$, *i. e.* if the Circuits of their Base be equal, they will be one to another as $\frac{1}{2} P : \frac{1}{2} p$, or as P to p .

P R O P. IX.

75. *Prisms* of the same, or an equal Base, are one to another as their Altitudes.

For $\left\{ \begin{array}{l} 1 \mid \frac{1}{2} ANBP = \\ 2 \mid \frac{1}{2} anbp = \frac{1}{2} aNBP, \text{ per hyp.} \end{array} \right\} = \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \text{ per. ar. 48}$
 But $\left| 3 \mid \frac{1}{2} ANBP : \frac{1}{2} aNBP : : A : a, \text{ per 16. 6.} \right.$
 Q. E. D.

P R O P. X.

76. Like *Prisms*, are one to another in a triplicate proportion of their homologous Sides.

For $\left\{ \begin{array}{l} 1 \mid \frac{1}{2} P : \frac{1}{2} p : : A : a, \text{ per Hyp.} \\ 2 \mid ANB : anb : : A^2 : a^2, \text{ per 19. 6.} \\ 3 \mid \frac{1}{2} ANBP : \frac{1}{2} anbp : : A^3 : a^3. \text{ Q.E.D.} \end{array} \right.$
 $I \times 2$

P R O P.

P R O P. XI.

77. If *Prisms* be equal, their Bases and Altitudes will be reciprocally proportional.

For $\left| \begin{array}{l} 1 \\ 2 \end{array} \right| \frac{1}{2} ANBP = \frac{1}{2} a b n p, \text{ per Hyp.}$

But $\left| \begin{array}{l} 1 \\ 2 \end{array} \right| \frac{1}{2} NBP : \frac{1}{2} n b p :: a : A, \text{ per 16.6. Q.E.D.}$

C O R O L L A R I E S.

78. 1. If the *Prisms* be both Triangular, Pentagonal, &c. then will $N=n$, and $\frac{1}{2} NP : \frac{1}{2} n p :: a : A$.

79. 2. If the Perimeters of their Bases be equal, i. e. $NB=nb$; then is $\frac{1}{2} P : \frac{1}{2} p :: a : A$.

80. 3. If $P=p$, then is $\frac{1}{2} NB : \frac{1}{2} nb :: a : A$.

P R O P. XII.

81. If a *Cylinder* be cut by a plane parallel to its Base, or which is the same thing, if two *Cylinders* have the same, or equal Bases, they will be in proportion one to another as their Altitudes.

For $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} AR^2 e \\ ar^2 e = R^2 ea, \text{ per hyp.} \end{array} \right\} = \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \text{Cylind. Art. 49.}$
 But $\left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| AR^2 e : a R^2 e :: A : a, \text{ per 16.6.}$

P R O P. XIII.

82. *Cylinders* that have equal Altitudes, are one to another in the proportion of their Bases.

For $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} AR^2 e \\ ar^2 e = Ar^2 e, \text{ per hyp.} \end{array} \right\} = \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \text{Cyl. per Ar. 49}$
 But $\left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| AR^2 e : Ar^2 e :: R^2 e : r^2 e.$

P R O P.

P R O P. XIV.

83. Like *Cylinders*, are in a triplicate Proportion of the Diameters of their Bases.

For $\left\{ \begin{array}{l} 1 \mid R^2 e : r^2 e :: R^2 : r^2, \text{ per 16. 6.} \\ 2 \mid A : a :: R : r, \text{ per hyp. and Art. 14.} \\ \text{But } 1 \times 2 \mid 3 \mid AR^2 e : ar^2 e :: R^3 : r^3. \text{ Q.E.D.} \end{array} \right.$

P R O P. XV.

84. If *Cylinders* be equal, their Bases and Altitudes will be reciprocally proportional.

For $\left\{ \begin{array}{l} 1 \mid AR^2 e = ar^2 e, \text{ per hyp.} \\ \text{But } 1 \mid 2 \mid A : a :: r^2 e : R^2 e. \end{array} \right.$

SCHOLIUM.

Thus have I given the Propositions belonging to these three Bodies severally, and shewed all along what Factors the Solidities, &c. are composed of, because I thought it would inform the Judgment better, and bring the Mind into a more familiar Acquaintance with all of them, than a more general, and abstracted Method could do. But now, for the sake of the more Curious, I will give some of them, as they may be deduced from the Propositions of the *Prism* only; but before I do that, I must observe,

85. 1. That all *Parallelepipedons*, are only Square, or (if I may so say) oblong *Prisms*; wherein $N = 4$, and $P = \frac{1}{2} L$, and so the Expression for the *Prism*, viz. $\frac{1}{2} ANBP$, will become $\frac{1}{2} A \times 4 \times B \times \frac{1}{2} L = ABL$, the same with the Expression for the *Parallelepipedon*.

86. 2. All *Cylinders* are only *Prisms*, having an infinite Number of Sides, and in that Case $B=0$, and $N=2Re$ the Circumference of the *Cylinder's* Base, and $P=R$ the Radius of the Base, and so the Expression $\frac{1}{2} ANBP$, will be AR^2e , the Expression for the *Cylinder*. This being premised, I proceed to,

87. *Example 1.* In Proposition 8, we have, $\frac{1}{2}ANBP : \frac{1}{2}anbp :: \frac{1}{2}NBP : \frac{1}{2}nbp$, and supposing $N=4$, and $P=\frac{1}{2}L$, it will be $ABL : abl :: BL : bl$, the same as in Proposition 3.

88. Again, suppose $B=0$, then is $N=2Re$, and $P=R$, &c. and so we shall have $AR^2e : ar^2e :: R^2e : r^2e$, as in Proposition 13.

89. *Example 2.* In Prop. 10, it is $\frac{1}{2}ANBP : \frac{1}{2}anbp :: A^3 : a^3$; and supposing $N=4$, and $P=\frac{1}{2}L$, it will be $ABL : abl :: A^3 : a^3$, as Prop. 4.

90. And if $B=0$, and $N=2Re$, and $P=R$, then it will be $AR^2e : ar^2e :: R^3 : r^3$, as in Prop. 14.

I might exemplify this Method in all the preceding Propositions; but this Specimen if well understood, will enable the young *Geometer* to do that himself.

P R O P. XVI.

Fig. 8.

91. Every *Prism*, as $ABCDEF$, having a triangular Base, may be divided into three *Pyramids*, $FDEC$, $CBFA$, and $FADC$.

Draw the Diagonals FC , FD , AC , which will bisect the *Prism* (p. 13.1.) now the *Pyramid* $FDEC = \text{Pyramid } CBFA$, having equal Bases and Altitudes;

tudes ; also for the same Reason the *Pyramid* $FACB = \text{Pyramid } FADC$; therefore the three *Pyramids* $FDCE$, $CBFA$, and $FADC$, are equal, *per Art.* 54.

SCHOLIUM.

92. Hence it is plain, that every *Pyramid* is the third Part of the *Prism* that hath the same Base and Altitude with it.

For, imagine any polygonous *Prism*, and *Pyramid*, whose Bases are equal, and like, and Height the same : Divide the first into triangular *Prisms*, and the other into triangular *Pyramids* ; then all the parts of the *Prism*, shall be treble all the parts of the *Pyramid*, and consequently the whole *Prism* is treble the whole *Pyramid*.

COROLLARIES.

93. Hence it evidently followeth, that all that hath been said of *Prisms*, is equally true of *Pyramids* ; for what proportion any Whole, hath to any other Whole, the same proportion hath any aliquot Part of the first, to a like aliquot Part of the second.

94. Hence also the Solidity of all *Pyramids* is easily found ; for if $\frac{1}{2} ANBP$, be the Expression for the Solidity of the *Prism*, as it is in *Art.* 48 ; then $\frac{1}{6} ANBP$ must be the Expression for the Solidity of a *Pyramid*, having the same Base and Altitude : *i. e.* the Solidity of all *Pyramids* is found by multiplying the Area of the Base, into $\frac{1}{3}$ part of the Altitude.

D



95. Hence

95. Hence also it may be demonstrated, that a *Cylinder* is treble of a *Cone*, having the same Base and Altitude.

For a *Cylinder* is only a *Prism*, having a multangular Base, and a *Cone* is also a *Pyramid*, having a multangular Base: But all *Prisms* are treble the *Pyramids* of the same Base and Altitude, therefore, &c.

96. All that hath been demonstrated of *Cylinders*, is equally true of *Cones*, they being $\frac{1}{3}$ part of their circumscribing *Cylinders*.

97. The Solidity of a *Cone* is hence easily found; for if AR^2e be the Expression for the Solidity of a *Cylinder*, as it is already proved to be, then $\frac{1}{3}AR^2e$ will be the Expression for the Solidity of a *Cone*, having the same Base and Altitude; *i. e.* the Solidity of a *Cone* is found by multiplying the Area of the Base into $\frac{1}{3}$ of the Altitude.

98. From hence, and from *Art.* 94, it is evident, that the Solidity of all Bodies, which may be conceived to be composed of *Cones* or *Pyramids*, as the *Sphere*, *Octaedron*, *Dodecaedron*, and *Icosaedron*, &c. may be easily found, their Areas being first known: For the Area of any of these Bodies, is only the Aggregate of as many circular, or polygonous Planes, as are the Number of *Cones*, or *Pyramids* which constitute the Body: But the Solidity of a *Cone* or *Pyramid*, is found by multiplying the Area of its Base, into $\frac{1}{3}$ part of the Height. And therefore, the Solidity of all the *Cones*, or *Pyramids*, *i. e.* the Solidity of the Body, is found, by multiplying the Sum of all the Areas, *i. e.* the Area of the Body, by $\frac{1}{3}$ part of the Perpendicular from the Center of the Body to the middle of one of the Planes, which is the common Height of all the *Cones*, or *Pyramids*.

99. The

99. The Converse of this is also hence evident, viz. That the Solidity of such a Body being known, the Area is thus found ; Divide the Solidity by $\frac{1}{3}$ part of the said Perpendicular, and the Quotient is the Area of the Body.

P R O P. XVII.

100. Every *Sphere*, is equal to two thirds of its circumscribing *Cylinder*.

Let the Square ad , Quadrant cbd , and the right-angled Triangle abd , be supposed all three to revolve round the Line bd as an Axis, then will the *Square* generate a *Cylinder*, the Quadrant an Hemisphere, and the Triangle a *Cone*, all of the same Base and Altitude.

Then | 1 | $\overline{eb}^2 (= \overline{fd}^2) = \overline{fb}^2 + \overline{bd}^2$, per 47.1.
 But | 2 | $\overline{bd}^2 = \overline{gb}^2$, per Figure.
 1, 2 | 3 | $\overline{eb}^2 = \overline{fb}^2 + \overline{gb}^2$, per Substitution,
 And so | 4 | $\odot eb = \odot fb + \odot gb$, per 31. 6.

Therefore if you take the Circle fb from both Sides, there will remain the Circle gb equal to the Ring described by the Motion of ef ; and thus it will always be wherever you draw the Line eb , or im , &c. therefore the Aggregate of all the Rings made by the Revolution of all the ef s, must be equal to that of all the gb 's, i. e. the Dish-like Solid formed by the Revolution of the Rings, will be equal to the *Cone*, formed by the Revolution of the gb 's, which are the Elements of the Triangle abd , i. e. the Dish-like Solid, will be as the *Cone* is, $\frac{1}{3}$ part of the circumscribing *Cylinder*; and consequently the Hemisphere must be $\frac{2}{3}$, therefore the Sphere is $\frac{2}{3}$ of the circumscribing *Cylinder*.

COROLLARIES.

101. Hence the Solidity of a Sphere is found: For if AR^2e be the Expression for the Solidity of a Cylinder, as it is in *Art.* 49. and if the Solidity of a Sphere be $\frac{2}{3}$ of its circumscribing Cylinder, as in the last; it must follow, that $\frac{2}{3}AR^2e$, will be the Expression for the Solidity of the Sphere. But in a Sphere $A=2R$, and so the Expression will be $\frac{4R^3e}{3}$, by Substitution.

102. If a Sphere, and a Cone, be both inscribed in the same Cylinder; the Sphere will be double the Cone; for the Solidity of the Sphere is expressed by $\frac{4R^3e}{3}$, per *Art.* 101; and the Solidity of a Cone by $\frac{2}{3}R^3e$, per *Art.* 97. But $\frac{4R^3e}{3} : \frac{2R^3e}{3} :: 2 : 1$, or double.

103. Hence, and from *Art.* 99, the Expression for the Area or Superficies of a Sphere is found to be $4R^2e$, *i. e.* four times the Area of its greatest Circle.

$$\text{For } \left. \begin{array}{l} 1 \mid \frac{4R^3e}{3} = \text{Solidity} \\ 1 \div \frac{R}{3} \mid 2 \mid 4R^2e = \text{Area} \end{array} \right\} \begin{array}{l} \text{of the Sphere,} \\ \text{per Art.} \end{array} \left\{ \begin{array}{l} 101 \\ 99 \end{array} \right.$$

That is, either multiply the Area of the greatest Circle of the Sphere by 4, or multiply the Square of the Diameter by (*e*) the common Ratio, and the Product will be the Area of the Sphere.

P R O P. XVIII.

104. Spheres are in a Triplicate Proportion, or in Proportion one to another as the Cubes of their Diameters.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{4 R^3 e}{3} = \right. \\ 2 \left| \frac{4 r^3 e}{3} = \right. \end{array} \right\} \text{Solidity of} \left\{ \begin{array}{l} \text{one} \\ \text{other} \end{array} \right\} \text{Sphere per} \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{4 R^3 e}{3} : \frac{4 r^3 e}{3} :: R^3 : r^3, \text{ per 16.6.} \right. \end{array} \right\} \text{Art. 101.} \end{array}$$

C O R O L L A R Y.

105. From this, and *Art.* 66, 76, 83, 93, 96, it is evident, that all similar Solids are one to another, as the Cubes of their homologous Sides.

P R O P. XIX.

106. The Superficies of all similar Solids, are one to another in Proportion, as the Squares of their homologous Sides.

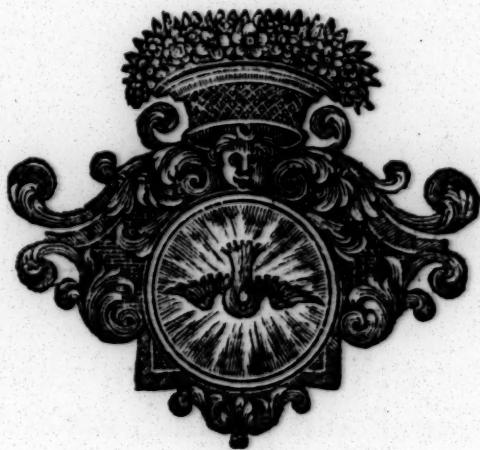
For all similar Solids, are bounded by an equal Number of similar Plains; which are reduceable to an equal Number of similar Triangles; and the Areas of similar Triangles are as the Squares of their homologous Sides, *per* 19. 6. but the Wholes are as their like Parts, Therefore, &c.

C O R O L -

COROLLARY.

107. All similar Solids, inscribed in Spheres, have their Superficies in Proportion one to another, as the Squares of the Radius's of their circumscribing Spheres.

For the Sides of the Solids are to one another as the Radius's of the circumscribing Spheres, *per* 4. 6. and the Superficies of similar Solids are one to another, as the Squares of their homologous Sides, *per Art.* 106, therefore, &c. *per* 11.5.





ELEMENTS

O F

SOLID GEOMETRY,

ALGEBRAICALLY Demonstrated.

BOOK II.

Of the PLATONIC BODIES.

CHAPTER I.

DEFINITION.



THE Regular, or Platonic Bodies, are such whose Surfaces are composed of regular and equal Figures, and whose Angles are all equal.

There are only five Bodies that can correspond to this Definition, viz. the *Tetraedron*, *Hexaedron*, *Octaedron*, *Dodecaedron*, and *Icosaedron*.

Section I. *Of the* TETRAEDRON.

A *Tetraedron* is a Pyramid, comprehended under four equal, and Equilateral Triangles; so that its Base is equal to each Side.

Generation.

Fig. 12. *Generation.* Suppose an Equilateral Triangle EFG , to be inscribed in a Circle, whose Center is H ; from whence having drawn the Radii HE , HF , HG , imagine them to be lifted up, together with their common Center H , and lengthened out, so that the point H ascending perpendicularly, you have the Lines IE , IF , IG , equal to the Lines EF , FG , GE ; after this way there will be a Space generated, consisting of four equal and Equilateral Triangles, which is called a *Tetraedron*.

L E M M A.

Fig. 13. 108. If in a Circle, there be inscribed an equilateral Triangle ABC , the part DG , cut off the Diameter by the Side of the Triangle, is one fourth part of the Diameter, and consequently AG is three fourth parts thereof.

For draw BD and BO , then $BD=BO$, (*per* 15. 4.) and BG common; and the Angles BGO , BGD , right, and therefore equal, and so the Triangles BGO , BDG , will be similar and equal, (*per* 4. 1.) and therefore $DG=GO=\frac{1}{2}DO=\frac{1}{4}AD$, and consequently $AG=\frac{3}{4}AD$.

P R O P. I.

109. The Diameter of a Sphere, is in Power to the Side of a *Tetraedron* inscribed in it, as 3 to 2.

For

For the Diameter of the circumscribing Sphere, put $2R$, and for the Side of the *Tetraedron* put a .

	1	$a^2 - \frac{a^2}{4} = \frac{3a^2}{4} = \overline{EL}^2$ per 47. 1.	Fig. 12.
$1 \text{ } \omega \times \frac{2}{3}$	2	$\frac{2}{3} \sqrt{\frac{3a^2}{4}} = HE$, per Art. 108.	
But	3	$2R : a :: a : \frac{a^2}{2R} = HI$, per 8. 6.	
	4	$a^2 - \frac{a^4}{4R^2} (= \overline{EL}^2 - \overline{IH}^2) = EH$. p. 47. 1.	
$2 \odot 2$	5	$\frac{a^2}{3} = \overline{EH}^2$	
$4=5$	6	$a^2 - \frac{a^4}{4R^2} = \frac{a^2}{3}$	
6 Red	7	$8R^2 = 3a^2$, i.e. $4R^2 : a^2 :: 3 : 2$.	

THEOREM.

Multiply the Square of the Diameter of any Sphere by 2, and divide that Product by 3, the Square Root of that Quotient will be the Side of the inscribed *Tetraedron*.

PROP. II.

110. The Diameter of the Sphere, is in power to the Diameter of the Circle encompassing the Triangle of a *Tetraedron* inscribed in it, as 9 to 8.

E

For

Elements of SOLID GEOMETRY

For the Diameter of this Circle put d ,

$$\begin{array}{lcl}
 \text{Then} & \left\{ \begin{array}{l} 1 \quad 2 \quad \sqrt{\frac{a^2}{3}} = \sqrt{\frac{4a^2}{3}} = d \text{ p. Step. 5.} \\ 2 \quad a^2 = \frac{8R^2}{3} \text{ p. Step. 7.} \end{array} \right. & \left. \vphantom{\begin{array}{l} 1 \quad 2 \quad \sqrt{\frac{a^2}{3}} = \sqrt{\frac{4a^2}{3}} = d \text{ p. Step. 5.} \\ 2 \quad a^2 = \frac{8R^2}{3} \text{ p. Step. 7.} \end{array}} \right\} \text{ of the last.} \\
 1, 2 & 3 \quad \sqrt{\frac{32R^2}{9}} = d = EN, & \\
 3 \odot 2 & 4 \quad \frac{32R^2}{9} = d^2, \text{ i. e. } 4R^2 : d^2 :: 9 : 8 &
 \end{array}$$

PROP. III.

III. The Diameter of a Sphere, is in power to the perpendicular Height of a *Tetraedron* inscribed in it, as 9 to 4.

For HI , the perpendicular Height put h .

$$\begin{array}{lcl}
 \text{Then} & \left\{ \begin{array}{l} 1 \quad \frac{a^2}{2R} = b \\ 2 \quad a^2 = \frac{8R^2}{3} \end{array} \right\} \text{ per Step. } & \left\{ \begin{array}{l} 3 \\ 7 \end{array} \right\} \text{ Ar. 109} \\
 1, 2 & 3 \quad \frac{4R}{3} = b. & \\
 3 \odot 2 & 4 \quad \frac{16R^2}{9} = b^2, \text{ i. e. } 4R^2 : b^2 :: 9 : 4. &
 \end{array}$$

COROLLARIES.

112. It is evident from the third Step of the last, that HI , the Height of the *Tetraedron*, is equal to two thirds of the Diameter of the Sphere.

113. The Diameter EN of the Circle encompassing the Triangle of a *Tetraedron*, is in power to the Height HI , as 2 to 1.

For

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \left\{ \begin{array}{l} 4 R^2 : d^2 :: 9 : 8 \\ 4 R^2 : b^2 :: 9 : 4 \\ d^2 : b^2 :: 8 : 4 :: 2 : 1. \end{array} \right\} \text{per Art.}$

But

110.
111.

P R O P. IV.

114. The Expression for the Area of a *Tetraedron*, in the Terms of the Side is $a^2 \sqrt{3}$.

$\begin{array}{l} 1 \times \frac{a}{2} \\ 2 \times \frac{a}{4} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \left\{ \begin{array}{l} \frac{a}{2} \sqrt{3} = EL. \\ \frac{a}{2} \times \frac{a}{2} \sqrt{3} = \frac{a^2}{4} \sqrt{3} = \text{Tri. } EFG \text{ p. Art. 28.} \\ a^2 \sqrt{3} = \text{Area.} \end{array} \right.$

103.

T H E O R E M.

Multiply the Square of the Side of a *Tetraedron* by the Square Root of 3 = 1.7320508, &c. and that Product will be the Area.

115. The Expression for the Area of a *Tetraedron* in the Terms of the Diameter is $\frac{8 R^2}{3} \sqrt{3} =$

$\frac{2}{\sqrt{3}} \times 4 R^2.$

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \left\{ \begin{array}{l} a^2 = \frac{8 R^2}{3} \\ a^2 \sqrt{3} = \text{Area per laft.} \\ \frac{8 R^2}{3} \sqrt{3} = \frac{3}{\sqrt{3}} \times 4 R^2 \end{array} \right.$

109.

T H E O R E M.

Multiply the Square of the Diameter of the Sphere by 2, and divide that Product by 1.7320508, &c. and that Quotient will be the Area of the inscribed *Tetraedron*.

E 2

P R O P.

For	{	1	$4R^2 = \frac{3a^2}{2}$ per Art.	109.
1×2		2	$\frac{8R^2}{3} = a^2$	
2×2		3	$\frac{2R\sqrt{2}}{\sqrt{3}} = a$	
2×3		4	$\frac{16R^3\sqrt{2}}{3\sqrt{3}} = a^3$	
But	{	5	$\frac{a^3}{6\sqrt{2}} = \text{Solidity.}$	116.
4, 5		6	$\frac{8R^3}{9\sqrt{3}} = \text{Solidity, per Substitution.}$	

THEOREM.

Divide the Cube of the Diameter, by 9 times the Square Root of 3, *i. e.* 15.588457, &c. and the Quotient will be the Solidity.

COROLLARY.

118. The Solidity of a *Tetraedron* to its Area, is as $\frac{R}{9}$ to 1, or as R to 9.

For	{	1	$\frac{8R^2}{\sqrt{3}} = \text{Area.}$	115.
		2	$\frac{8R^3}{9\sqrt{3}} = \text{Solidity}$	117.
		3	$\frac{8R^3}{9\sqrt{3}} : \frac{8R^2}{\sqrt{3}} :: \frac{R}{9} : 1 :: R : 9.$	

PROP.

P R O P. VI.

119. The Diameter of a Sphere circumscribing a *Tetraedron*, is in Power to the Diameter of a Sphere whose Area is equal to the Area of the *Tetraedron*, as $e : \frac{1}{\sqrt{3}}$.

For the Diameter of the Sphere, whose Area is equal to the Area of the *Tetraedron*, put $2r$.

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{8 R^2}{\sqrt{3}} = \right. \\ 2 \left| 4 r^2 e = \right. \end{array} \right\} \text{Area of the } \left\{ \begin{array}{l} \text{Tetraed.} \\ \text{Sphere} \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 115. \\ 31. \end{array} \right.$$

$$1, 2 \quad \left| \begin{array}{l} 3 \left| 4 R^2 : 4 r^2 :: e : \frac{1}{\sqrt{3}} :: 3,1415926 : ,5773505, \&c \end{array} \right.$$

P R O P. VII.

120. The Diameter of a Sphere circumscribing a *Tetraedron*, is in Power to the Diameter of a Sphere whose Solidity is equal to the Solidity of the *Tetraedron*, as $3e\sqrt{3} : 2$.

For the Diameter of this Sphere put $2x$.

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{8 R^3}{9\sqrt{3}} = \right. \\ 2 \left| \frac{4x^3 e}{3} = \right. \end{array} \right\} \text{Solidity of the } \left\{ \begin{array}{l} \text{Tetraed} \\ \text{Sphere} \end{array} \right\} \text{p. Art. } \left\{ \begin{array}{l} 117 \\ 101 \end{array} \right.$$

$$\text{But } \left\{ \begin{array}{l} 3 \left| \frac{8 R^3}{9\sqrt{3}} = \frac{4x^3 e}{3} = \frac{8x^3 e}{6} \text{ per Hypothesis} \right. \end{array} \right.$$

$$\text{therefore } \left| \begin{array}{l} 4 \left| 8 R^3 : 8x^3 :: \frac{e}{6} : \frac{1}{9\sqrt{3}} :: 3e\sqrt{3} : 2. \right. \end{array} \right.$$

COROL-

COROLLARY.

121. The Diameter of a Sphere, whose Solidity is equal to the Solidity of a given *Tetraedron*, is in power to the Diameter of a Sphere, whose Superficies is equal to the Superficies of the said *Tetraedron*, as R to $3x$.

For	{	1	$8R^3 : 8x^3 :: 3e\sqrt{3} : 2 :: e : \frac{2}{3\sqrt{3}}.$	p. 119.
		2	$4R^2 : 4r^2 :: e : \frac{1}{\sqrt{3}}.$	p. 120.
$1 \div d$		3	$4R^2 : \frac{4x^2}{R} :: e : \frac{1}{3x\sqrt{3}}, p. 11.5.$	
2, 3		4	$\frac{4x^2}{R} : 4r^2 :: \frac{1}{3x\sqrt{3}} : \frac{1}{\sqrt{3}}, per 11.5$	
4...		5	$4x^2 : 4r^2 :: R : 3x.$	

Section II. Of the HEXAEDRON.

DEFINITION.

THE *Hexaedron*, or *Cube*, hath its Superficies composed of six equal Squares, like a Die which we play with. Fig. 7 N. 2.

Generation. If the Plane of a Square AF , be conceived to move along EC , another Square equal to it self; and always in a Position parallel to it self; there will be generated a Body, or Space, contained under six equal Squares, as $ABHCGFED$, which is called a *Cube* or *Hexaedron*.

P R O P.

PROP. I.

122. The Diameter of a *Sphere* is in Power to the side of a *Cube* inscribed in it, as 3 to 1.

Fig. 7.

All being drawn as in the Figure, we have $DE = DH = EA = HA = BC$, &c. for the sides of the *Cube*, which we call x , and AC the Diameter of the *Sphere* call $2R$.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \mid \overline{AF}^2 = \overline{AH}^2 + \overline{HF}^2 = 2x^2, \text{ per 47.1} \\ 2 \mid \overline{AC}^2 (= 4R^2) = \overline{AF}^2 + \overline{FC}^2 = 3x^2 \text{ p. di:} \end{array} \right. \\ 2 \text{ Analo. } \left\{ \begin{array}{l} 3 \mid 4R^2 : x^2 :: 3 : 1. \end{array} \right. \end{array}$$

PROP. II.

123. The Diameter of the *Sphere* circumscribing a *Cube*, is in Power, to the Diameter of a Circle encompassing the Square of the *Cube*, as 3 to 2.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \mid \overline{AF}^2 = 2x^2 = \text{square of the Diam. of the} \\ 2 \mid 4R^2 = 3x^2, \text{ per last (Circle, 47.1.} \end{array} \right. \\ \text{But } \left\{ \begin{array}{l} 3 \mid 3x^2 (= 4R^2) : 2x^2 :: 3 : 2, \text{ p. 16. 6.} \end{array} \right. \end{array}$$

COROLLARIES.

124. The side of a *Tetraedron*, is in Power to the Diameter of a Circle encompassing the square of a *Cube* inscribed in the same *Sphere*, in a Proportion of Equality.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \mid 4R^2 : a^2 :: 3 : 2 \\ 2 \mid 4R^2 : x^2 :: 3 : 2 \end{array} \right\} \text{ p. Art. } \left\{ \begin{array}{l} 109. \\ 122. \end{array} \right. \\ 1, 2 \quad \left\{ \begin{array}{l} 3 \mid 4R^2 : 4R^2 : a^2 : x^2, \text{ or equal.} \end{array} \right. \end{array}$$

125. The Diameter of a *Sphere*, is equal in Power to the side of a *Cube*, and *Tetraedron* inscribed in it taken together.

For

$$\text{For } \left\{ \begin{array}{l} 1 \left| a^2 = \frac{8R^2}{3} \right. \\ 2 \left| x^2 = \frac{4R^2}{3} \right. \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 109. \\ 122. \end{array} \right.$$

$$1+2 \left| 3 \left| a^2 + x^2 = \frac{8R^2 + 4R^2}{3} = 4R^2. \right. \right.$$

P R O P. III.

126. The Expression for the Superficies of a *Cube*, in the Terms of the side is $6x^2$.

$$\text{For } 1 \times 6 \left| \begin{array}{l} 1 \left| x^2 = \text{Area of one plane,} \\ 2 \left| 6x^2 = \text{whole Superficies,} \end{array} \right. \right\} \text{ per Art. } \left\{ \begin{array}{l} 27. \\ 37. \end{array} \right.$$

T H E O R E M.

Multiply the square of the side by 6, and that Product will be the Superficies.

127. The Expression for the Superficies of a *Cube*, in the Terms of the Diameter of the circumscribing *Sphere* is $8R^2$.

$$\text{For } \left| 1 \left| x^2 = \frac{4R^2}{3} \right. \right. \text{ per Art. 122.}$$

$$\text{And } 1 \times 6 \left| 2 \left| 6x^2 = 8R^2 = \text{Superficies. 37.} \right. \right.$$

T H E O R E M.

Multiply the square of the Diameter by 2, and that Product will be the Superficies of the *Cube*.

P R O P. IV.

128. The Expression for the Solidity of a *Cube* in the terms of its side is x^3 .

F

This

This is evident from *Art.* 50.

129. The Expression for the Solidity of a *Cube* in the Terms of its circumscribing *Sphere* is

$$\frac{8 R^3}{3\sqrt{3}}.$$

$$\begin{array}{l} \text{For} \\ 1 \text{ w } 2 \\ 1 \times 2 \end{array} \left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} x^2 = \frac{4 R^2}{3} \\ x = \frac{2 R}{\sqrt{3}} \\ x^3 = \frac{8 R^3}{3\sqrt{3}} \end{array} \begin{array}{l} 122. \\ \\ = \text{Solidity } 50. \end{array}$$

THEOREM.

Divide the Cube of the Diameter of a *Sphere*, by three times the square Root of 3, *viz.* 5.1961524, and the Quotient will be the Solidity of the inscribed Cube.

COROLLARIES.

130. The Solidity of a *Cube* is to the Solidity of a *Tetraedron* inscribed in the same *Sphere*, as 3 to 1.

$$\begin{array}{l} \text{For} \\ \text{But} \end{array} \left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} \frac{8 R^3}{3\sqrt{3}} \\ \frac{8 R^3}{9\sqrt{3}} \\ \frac{8 R^3}{3\sqrt{3}} : \frac{8 R^3}{9\sqrt{3}} \end{array} = \left\{ \begin{array}{l} \text{Solidity} \\ \text{of the} \end{array} \right\} \left\{ \begin{array}{l} \text{Cube} \\ \text{Tetraedron} \end{array} \right\} \left\{ \begin{array}{l} p. \text{ Ar. } \\ \\ \end{array} \right\} \begin{array}{l} 129. \\ 117. \end{array}$$

:: 3 : 1, per 16.6.

131. The Solidity of a *Cube*, is to the Solidity of a *Tetraedron* inscribed in the same *Sphere*, as the square of the Diameter of the *Sphere*, is to the square of the side of the *Cube*.

For

Algebraically Demonstrated.

35

For $\left\{ \begin{array}{l} 1 \left| \frac{8R^3}{3\sqrt{3}} : \frac{8R^3}{9\sqrt{3}} :: 3 : 1. \right. \\ 2 \left| \frac{4R^2}{8R^3} : \frac{x^2}{8R^3} :: 3 : 1. \right. \end{array} \right\} \text{per Art.} \left\{ \begin{array}{l} 130. \\ 122. \end{array} \right.$

But $3 \left| \frac{8R^3}{3\sqrt{3}} : \frac{8R^3}{9\sqrt{3}} :: 4R^2 : x^2, \text{ per 11.5.} \right.$

P R O P. V.

132. The Diameter of a *Sphere* circumscribing a *Cube*, is in power to the Diameter of a *Sphere* whose Superficies is equal to the Superficies of the *Cube*, as the Ratio of the Diameter of a Circle, to its Circumference, is to 2, *i. e.* as *e* to 2.

For the Diameter of this *Sphere* put $2r$.

For $\left\{ \begin{array}{l} 1 \left| 8R^2 \right. \\ 2 \left| 4r^2e \right. \end{array} \right\} = \text{Sup. of the } \left\{ \begin{array}{l} \text{Cube,} \\ \text{Sphere, \&c.} \end{array} \right. \begin{array}{l} 127. \\ 31. \end{array}$

$1 = 2$ $3 \left| 8R^2 = 4r^2e, \text{ per Hypoth.} \right.$

3 Analo $4 \left| 4R^2 : 4r^2 :: e : 2, \text{ per 16.6.} \right.$

P R O P. VI.

133. The Diameter of a *Sphere* circumscribing a *Cube*, is in power to the Diameter of a *Sphere* whose Solidity is equal to the Solidity of the *Cube*, as the Ratio of the Diameter of a Circle to its Circumference multiplied into the square Root of 3, is to 2.

F 2

For

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \left| \begin{array}{l} 8R^3 \\ 3\sqrt{3} \\ 4x^3e \\ 3 \end{array} \right. \\ 2 \end{array} \right\} = \text{Solidity of the} & \left\{ \begin{array}{l} \text{Cube. } 129. \\ \text{Sphere. } 104. \end{array} \right. \\
 1 = 2 & 3 \left| \begin{array}{l} 8R^3 \\ 3\sqrt{3} \end{array} \right. & = \frac{4x^3e}{3} \left(= \frac{8x^3e}{6} \right) \text{ per Hyp.} \\
 3 \text{ Analo} & 4 \left| \begin{array}{l} 8R^3 : 8x^3 :: e\sqrt{3} : 2. \end{array} \right. &
 \end{array}$$

COROLLARY.

134. The Diameter of a Sphere whose Solidity is equal to the Solidity of a Cube, is in power to the Diameter of a Sphere whose Superficies is equal to the Superficies of the Cube, as $2R$ is to $2x\sqrt{3}$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \left| \begin{array}{l} 8R^3 : 8x^3 :: e\sqrt{3} : 2 :: e : \frac{2}{\sqrt{3}}. \\ 2 \left| \begin{array}{l} 4R^3 : 4r^2 :: e : 2. \text{ p. } 132. \\ 3 \left| \begin{array}{l} 4R^2 : \frac{4x^2}{R} :: e : \frac{2}{x\sqrt{3}}. \\ 4 \left| \begin{array}{l} \frac{4x^2}{R} : 4r^2 :: \frac{2}{x\sqrt{3}} : 2, \text{ per } 11.5. \\ 5 \left| \begin{array}{l} 4x^2 : 4r^2 :: 2R : 2x\sqrt{3}. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \\
 1 \div d & & \\
 \text{But} & & \\
 4 \text{ red.} & &
 \end{array} \right. \quad 133
 \end{array}$$

Section III. Of the OCTAEDRON.

DEFINITION.

THE *Octaedron* is bounded by eight, equal, and equilateral Triangles, and is two square *Pyramids* put together at their Bases.

Gene-

Generation. The Generation of the *Octaedron* is much like that of the *Tetraedron*; for by a mental raising of *C* the Center of the Square *ABDE* inscribed in a Circle, together with the Semidiameters *CA, CB, CD, CE*, until being more and more extended, they at length become the lines *AF, BF, DF, EF*, all equal among themselves, and to the sides of the Square *AB* or *BD*; and it's manifest, that by the like Extension conceived to be made downwards to *G*, there will be formed eight equal, and Equilateral Triangles, which will all concur in two opposite points *F* and *G*.

P R O P. I.

135. The Diameter of the Sphere, is in power to the side of an *Octaedron* inscribed in it, as 2 to 1.

For the Diameter of the Sphere *FG*, put $2R$, and for the side of the *Octagon* $AG = AF$, &c. put z .

For $\left| \begin{array}{l} 1 \\ 2 \end{array} \right| \overline{AG}^2 + \overline{AF}^2 (=2z^2) = \overline{FG}^2 (=4R^2) \text{ p. 47.1}$
 1 Analo $\left| \begin{array}{l} 1 \\ 2 \end{array} \right| 4R^2 : z^2 :: 2 : 1.$

C O R O L L A R Y.

136. The Diameter of a Sphere, is to the side of an *Octaedron*, as the Diameter of a Circle encompassing the Triangle of a *Tetraedron*, is to the height of the *Tetraedron*.

For $\left\{ \left| \begin{array}{l} 1 \\ 2 \end{array} \right| 4R^2 : Z^2 :: 2 : 1 \right\} \text{ per Art. } \left\{ \begin{array}{l} 135. \\ 113. \end{array} \right.$
 But $\left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| 4R^2 : Z^2 :: d^2 : b^2, \text{ per 11.5.}$

P R O P.

P R O P. II.

137. The Diameter of the Sphere, is in power to the Diameter of a Circle encompassing the Triangle of an *Octaedron*, as 3 to 2.

For the Diameter of this Circle put f .

	1	$\overline{AB}^2 - \overline{AH}^2 \left(= \frac{3Z^2}{4} \right) = \overline{BH}^2$; per 47.1.	
1 $\propto$ 2	2	$\frac{Z}{2} \sqrt{3} = BH$	108
2 $\times \frac{4}{3}$	3	$\frac{2Z}{3} \sqrt{3} = f$	
3 $\odot$ 2	4	$\frac{4Z^2}{3} = f^2$	
4 $\div \frac{2}{3}$	5	$2Z^2 = \frac{3f^2}{2} = 4R^2$, per Art. 135.	
5 Analo	6	$4R^2 : f^2 :: 3 : 2$.	

C O R O L L A R I E S.

138. The Diameter of a Circle encompassing the Triangle of an *Octaedron*, is equal to the sides of a *Tetraedron* inscribed in the same Sphere.

For {	1	$4R^2 : f^2 :: 3 : 2$.	137.
	2	$4R^2 : a^2 :: 3 : 2$.	109.
But	3	$f^2 : a^2 :: 2 : 2$, or equal,	per 11.5.

139. From this, and Art. 123. it appears, that if the *Tetraedron*, *Cube*, and *Octaedron*, be all inscribed in one and the same Sphere, the side of the *Tetraedron*, the Diameter of the Circle encompassing the Square of the *Cube*, and Diameter of the Circle encompassing the Triangle of an *Octaedron* are all equal.

P R O P.

P R O P. III.

140. The Perpendicular height of the Triangle of an *Octaedron*, is in power to the Diameter of the circumscribing Sphere, as 3 to 8.

For the perpendicular BH , put p .

$$\begin{array}{lcl}
 & \left| \begin{array}{l} 1 \quad \frac{Z}{2} \sqrt{3} = p, \text{ p. 108.} \\ 2 \quad \frac{3 Z^2}{4} = p^2 \\ 3 \quad 2 Z^2 = \frac{8 p^2}{3} = 4 R^2 \text{ (per Art. 135) i.e.} \\ \quad p^2 : 4 R^2 :: 3 : 8, \text{ per 16.6.} \end{array} \right. & \\
 1 \odot 2 & & \\
 2 \times 2, \&c. & &
 \end{array}$$

COROLLARIES.

141. The Diameter of a Circle encompassing the Triangle of an *Octaedron*, is in power to the Perpendicular height of the said Triangle, as 16 to 9.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \quad 4 R^2 : f^2 :: 3 : 2 :: 24 : 16 \\ 2 \quad 4 R^2 : p^2 :: 8 : 3 :: 24 : 9 \end{array} \right\} & \begin{array}{l} \text{per } \{ 137. \\ \text{Art. } \{ 140. \end{array} \\
 \text{But} & \left| \begin{array}{l} 3 \quad f^2 : p^2 :: 16 : 9, \text{ per 11.5.} \\ 4 \quad f : p :: 4 : 3, \text{ exactly agreeing with} \\ \quad \text{Art. 108} \end{array} \right. &
 \end{array}$$

142. The Diameter of a Circle encompassing the Triangle of a *Tetraedron*, is in power to the Diameter of a Circle encompassing the Triangle of an *Octaedron*, inscribed in the same Sphere, as 4 to 3.

For

$$\begin{array}{l} \text{For} \\ \text{But} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \middle| \begin{array}{l} 4R^2 : d^2 :: 9 : 8. \\ 4R^2 : f^2 :: 9 : 6. \\ d^2 : f^2 :: 8 : 6 :: 4 : 3, \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 110. \\ 137. \\ 115. \end{array} \right.$$

P R O P. IV.

143. The Expression for the Superficies of an *Octaedron* in the Terms of the Side, is $2Z\sqrt{3}$.

$$\begin{array}{l} 1 \times Z \\ 2 \times 8 \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \middle| \begin{array}{l} \frac{Z}{4} \sqrt{3} = \frac{1}{2} BH, \text{ per } 108. \\ \frac{Z^2}{4} \sqrt{3} = \text{Area of the Triangle } ABG, \text{ p. } 28. \\ 2 Z^2 \sqrt{3} = \text{Superficies of the } \textit{Octaedron}. \end{array} \right.$$

T H E O R E M.

Multiply the square of the Side of an *Octaedron*, by twice the Square Root of three, *i. e.* by 3.464-1016, and the Product will be the Superficies of the *Octaedron*.

144. The Expression for the Superficies of an *Octaedron* in the Terms of the Diameter of its circumscribing Sphere is $4R^2\sqrt{3}$.

$$\begin{array}{l} \text{For} \\ 2 \times \sqrt{3} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \middle| \begin{array}{l} 2 Z^2 \sqrt{3} = \text{Superficies} \\ 2 Z^2 = 4 R^2 \\ 4 R^2 \sqrt{3} = \text{Superficies.} \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 143. \\ 135. \end{array} \right.$$

T H E O R E M.

Multiply the Square of the Diameter of a Sphere, by the Square Root of 3, *i. e.* by 1.7320508, and the Product will be the Superficies of the inscribed *Octaedron*.

COROL-

COROLLARIES.

145. The Superficies of an *Octaedron*, is to the Superficies of a *Tetraedron* inscribed in the same Sphere, as 3 to 2.

$$\left\{ \begin{array}{l} 1 \\ 2 \end{array} \middle| \begin{array}{l} 4 R^2 \sqrt{3} \\ \frac{8 R^2 \sqrt{3}}{3} \end{array} \right\} = \text{Sup. of a } \left\{ \begin{array}{l} \text{Octaed.} \\ \text{Tetraed.} \end{array} \right\} \text{ per Art } \left\{ \begin{array}{l} 144 \\ 115 \end{array} \right.$$

$$\left\{ \begin{array}{l} 3 \end{array} \middle| 4 R^2 \sqrt{3} : \frac{8 R^2}{3} \sqrt{3} :: 3 : 2. \text{ p. 16. 6.} \right.$$

146. If an *Octaedron* and a *Tetraedron* be both inscribed in the same Sphere, the Superficies of the *Octaedron*, is to the Superficies of the *Tetraedron*, as the Square of the Diameter of the circumscribing Sphere, is to the Square of the Side of the *Tetraedron*.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \middle| \begin{array}{l} 4 R^2 \sqrt{3} : \frac{8 R^2}{3} \sqrt{3} :: 3 : 2 \\ 4 R^2 : a^2 :: 3 : 2. \end{array} \right\} \text{ p. Art } \left\{ \begin{array}{l} 145 \\ 109 \end{array} \right.$$

$$\text{But } \left\{ \begin{array}{l} 3 \end{array} \middle| 4 R^2 \sqrt{3} : \frac{8 R^2}{3} :: 4 R^2 : a^2, \text{ p. 11. 5.} \right.$$

147. If there be an *Octaedron*, a *Tetraedron*, and a *Cube*, all inscribed in the same Sphere, the Superficies of the *Octaedron*, is to the Superficies of the *Tetraedron*, as the Square of the Diameter of the circumscribing Square, is to the Square of the Diameter of a Circle encompassing the Square of the *Cube*.

G

For

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \left| 4R^2\sqrt{3} : \frac{8R^2}{3}\sqrt{3} :: 3 : 2 \right. \\ 2 \left| 4R^2 : 2x^2 :: 3 : 2. \right. \end{array} \right. \\ \text{But } \left\{ \begin{array}{l} 3 \left| 4R^2\sqrt{3} : \frac{8R^2}{3}\sqrt{3} :: 4R^2 : 2x^2. \right. \end{array} \right. \end{array}$$

P R O P. V.

148. The Expression for the Solidity of an *Octaedron* in the Terms of its Sides is $\frac{Z^3\sqrt{2}}{3}$.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \left| Z\sqrt{2} = 2R = FG \text{ Height both } Pyr. \quad 47.1 \\ 2 \left| Z^2 = \text{Area of their com. Base } AFDG \quad 27. \right. \end{array} \right. \\ 1 \times \frac{1}{3} \times 2 \left\{ \begin{array}{l} 3 \left| \frac{Z^3\sqrt{2}}{3} = \text{Solidity, per Art. 94.} \right. \end{array} \right. \end{array}$$

T H E O R E M.

Multiply the Cube of the Side of an *Octaedron*, by the Square Root of 2, *i. e.* by 1.41421, and divide that Product by 3, and that Quotient will be the Solidity.

149. The Expression for the Solidity of an *Octaedron* in the Terms of the Diameter of its circumscribing Sphere is $\frac{8R^3}{6}$.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \left| Z = R\sqrt{2}, \text{ per } 135. \\ 1 \times 2 \left| 2 \left| Z^2 = 2R^2. \right. \right. \\ 1 \times 2 \left| 3 \left| Z^3 = 2R^3\sqrt{2}. \right. \right. \end{array} \right. \\ 3 \times \frac{\sqrt{2}}{3} \left\{ \begin{array}{l} 4 \left| \frac{Z^3\sqrt{2}}{3} = \frac{4R^3}{3} = \frac{8R^3}{6} = \text{Sol. } 148. \right. \end{array} \right. \end{array}$$

T H E O-

THEOREM.

Divide the Cube of the Diameter of a Sphere by 6, and the Quotient will be the Solidity of the inscribed *Octaedron*.

COROLLARY.

150. The Solidity of a Sphere, is to the Solidity of its inscribed *Octaedron*, as e , the constant Ratio between the Diameter of a Circle and its Circumference, to 1.

For $\left\{ \begin{array}{l} 1 \frac{4R^3 e}{\sqrt{3}} = \\ 2 \frac{4R^3}{3} = \end{array} \right\} \text{Solidity of the } \left\{ \begin{array}{l} \text{Sphere} \\ \text{Octaed.} \end{array} \right\} \left\{ \begin{array}{l} \text{per} \\ \text{Ar} \end{array} \right\} \left\{ \begin{array}{l} 101 \\ 149 \end{array} \right\}$

But $\left| \begin{array}{l} 3 \frac{4R^3 e}{3} : \frac{4R^3}{3} :: e : 1, \text{ per } 16.6 :: \\ \text{Circumf.} : \text{Diameter.} \end{array} \right.$

PROP. VI.

151. The Diameter of a *Sphere* circumscribing an *Octaedron*, is in power to the Diameter of a *Sphere*, whose Superficies is equal to the Superficies of the *Octaedron*, as the Ratio of the Diameter of a Circle to its Circumference, is to the Square Root of 3.

$$\begin{array}{l}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left| \begin{array}{l} 4R^2\sqrt{3} \\ 4r^2e \end{array} \right\} = \text{Sup. } \left. \begin{array}{l} \text{Octaedron.} \\ \text{of the Sphere, \&c.} \end{array} \right\} \begin{array}{l} 144. \\ 31. \end{array} \\
 1=2 \quad \left. \begin{array}{l} 3 \\ 4 \end{array} \right\} \left| \begin{array}{l} 4R^2\sqrt{3} = 4r^2e, \text{ per Hypoth.} \\ 4R^2 : 4r^2 :: e : \sqrt{3}, \text{ per 16.6.} \end{array} \right.
 \end{array}$$

P R O P. VII.

152. The Diameter of a *Sphere* circumscribing an *Octaedron*, is in power to the Diameter of a *Sphere*, whose Solidity is equal to the Solidity of an *Octaedron*, as the Ratio of the Diameter of a Circle, &c. is to 1.

$$\begin{array}{l}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left| \begin{array}{l} \frac{8R^3}{6} \\ \frac{4x^3e}{3} \end{array} \right\} \text{Solidity of the } \left\{ \begin{array}{l} \text{Octaedron,} \\ \text{Sphere, \&c.} \end{array} \right\} \begin{array}{l} 149. \\ 101. \end{array} \\
 1=2 \quad \left. \begin{array}{l} 3 \\ 4 \end{array} \right\} \left| \begin{array}{l} \frac{8R^3}{6} = \frac{4x^3e}{3}, \text{ per Hypoth.} \\ 8R^3 : 8x^3 :: e : 1, \text{ per 16.6.} \end{array} \right.
 \end{array}$$

C O R O L L A R Y.

153. The Diameter of a *Sphere*, whose Solidity is equal to the Solidity of an *Octaedron*, is in power to the Diameter of a *Sphere*, whose Superficies is equal to the Superficies of the *Octaedron*, as $2R$ is to $2x\sqrt{3}$.

For

For	1	$8 R^3 : 8 x^3 :: e : 1$, p. 152
	2	$4 R^2 : 4 r^2 :: e : \sqrt{3}$. p. 151.
$1 \div d$	3	$4 R^2 : \frac{4 x^2}{2 R} :: e : \frac{1}{2 x}$.
But	4	$\frac{4 x^2}{2 R} : 4 r^2 :: \frac{1}{2 x} : \sqrt{3}$, per 11.5.
4 red	5	$4 x^2 : 4 r^2 :: 2 R : 2 x \sqrt{3}$.

Sect. IV. Of the DODECAEDRON.

DEFINITION.

THE *Dodecaedron*, is a Solid contained under Fig. 14. twelve equal, and equilateral Planes; consisting of twelve equal *Pyramids*, with pentagonal Bases, whose common Vertex is the Center of the circumscribing *Sphere*.

Generation. The Generation, or Construction of this Solid, may be thus, *viz.* Upon the side *AB* of a *Cube*, let there be set, or conceived to be set, two Angles eAB, eBA , each of 36 Degrees, then the Angle AeB being 108 Degrees, is the Angle of a Pentagon. On the other side of the Line *AB*, make the Angles ABc, BAd , each of 72 Degrees, and the Lines *Bc, Ad*, equal to *Be, Ae*, and join *cd*. Then we shall have a regular Pentagon, whose two Angles *A* and *B*, lying upon the side *AB* of the *Cube*, which will be extended beyond *EF*, the middle of the *Cube*, when lying along the same. Now in the other Figure, conceive this *Pentagon* to be moved, or carried up above the Line *AB*, as it were

were an Axis, until the side cd thereof, stand perpendicularly over the Line EF , the middle of the *Cube*, and another Pentagon equal and similar to it, be applied on the opposite side, and so ten other equal, and similar *Pentagons* upon the ten remaining sides of the *Cube*; then it will not be difficult to imagine a new Body will be circumscribed about the *Cube*, comprehended under twelve equal, and regular *Pentagons*, which is called a *Dodecaedron*.

P R O P. I.

154. The side of a *Dodecaedron*, is equal in Power to the greater Part of the side of the *Cube*, divided in mean and extream Proportion.

For, if to the side of the *Cube* AB , and to its upper Base $ABCD$, you conceive accommodated, or fitted a regular *Pentagon*, according to the Genesis of the *Dodecaedron*, and with the Interval Be , you describe the Arch ef , and conceive the straight Line ef to be drawn, the Triangles ABe , Aef , will be equiangular; for the Angles at A and B are equiangular, being each 36 Degrees, and AeB 108 Degrees, ef being drawn, the Angles Bef , Bfe , are each 72 Degrees, therefore Aef the remaining Angle will be 36 Degrees; wherefore, As AB : to Be ($=Bf$, (per *Const.*) so is Ae ($Be=Bf$) to Af ; therefore the side of the *Cube* AB is divided in mean and extream Proportion in f , and Be the side of the *Dodecaedron* is equal to Bf , the greater Part of AB , the side of the *Cube*.

P R O P.

P R O P. II.

155. The Diameter of a *Sphere*, is in power to the side of a *Dodecaedron* inscribed in it, as 2 to

$$1 - \frac{\sqrt{5}}{5}.$$

For the Diameter of the *Sphere* put $2R$, and x for the side of the *Cube*, and y for the greater Segment of x , or side of the *Dodecaedron*.

	1	$4R^2 = 3x^2$, per <i>Art.</i> 122.
100 2, &c.	2	$\sqrt[3]{\frac{4R^2}{3}} = x$.
	3	$y = \square$
	4	$\sqrt[3]{\frac{4R^2}{3}} - y = \square$ } part 1 of x .
Then	5	$\frac{4R^2}{3} - y\sqrt[3]{\frac{4R^2}{3}} = y^2$. per Hypoth.
$5 + y \frac{\sqrt{4R^2}}{3}$	6	$y^2 + y \sqrt[3]{\frac{4R^2}{3}} = \frac{4R^2}{3}$
$6 + \frac{R^2}{3}$	7	$y^2 + y \sqrt[3]{\frac{4R^2}{3}} + \frac{R^2}{3} = \frac{5R^2}{3}$.
7 00 2	8	$y + \sqrt[3]{\frac{R^2}{3}} = \sqrt[3]{\frac{5R^2}{3}}$
8 —	9	$y = \sqrt[3]{\frac{5R^2}{3}} - \sqrt[3]{\frac{R^2}{3}}$.
9 0 2	10	$y^2 = 2R^2 - \frac{2R^2}{3}\sqrt{5} = R^2 \times 2 - \frac{2\sqrt{5}}{3}$
10 x 2	11	$2y^2 = 4R^2 \times 1 - \frac{\sqrt{5}}{3}$.
11 Anal	12	$4R^2 : y^2 :: 2 : 1 - \frac{\sqrt{5}}{2}$.

Put

Put $b = 2 \frac{2\sqrt{5}}{3}$ then $y^2 = R^2 b$, and $b = 509.3333$, &c.

L E M M A.

Fig. 16. 156. The side DF , of a regular *Pentagon* $ABDFG$, is equal in power to the side of an *Hexagon*, and *Decagon* inscribed in the same Circle.

Make $DC = m$. The Angle DEF is 72 Deg. and the others in the Triangle at D and F , are each 54 Degrees; but FCG is also 54 Degrees, as subtending the Decagonal Arch FX , of 36 Degrees, and XG of 18 Degrees, also one half of it. Therefore the Triangles DCF , ICF , are equiangular, whence,

$$DF : m :: m : FI \text{ (p. 16.6)} = \frac{m^2}{DF}.$$

Secondly, in the Triangle DFX , the Angles at D , F , are equal, *per* 5. 1, and also the Angles at D , X , in the Triangle DXI are equal too; therefore the Triangles DFX , DXI , are equiangular;

therefore $DF : DX :: DX : \frac{DX^2}{DF} = DI$, the

whole Line $DF = \frac{m^2}{DF} + \frac{DX^2}{DF}$, therefore $DF^2 = m^2 + DX^2$.

L E M M A II.

157. If a *Pentagon* $ABDF$, whose side we will call y , be inscribed in a Circle, and if in the same Circle a *Decagon* be inscribed, the Diameter of the circum-

circumscribing Circle will be expressed by $y \times \sqrt{2 + 2\sqrt{2}}$

And the side of the Decagon by $y \times \sqrt{5 - \sqrt{5}}$. Fig. 16.

Let the Diameter of AX of the circumscribing Circle be called $2m$, and let DX be the side of the Decagon.

1	$y^2 = m^2 + \overline{DX}^2$, per Lemma antec.
2	$y^2 - m^2 = \overline{DX}^2$.
3	$2m \times HX = \overline{DX}^2$, per Cor. 2, 8, 6.
4	$y^2 - m^2 = 2m \times HX$.
5	$\frac{y^2 - m^2}{2m} = HX$.
6	$m - \frac{y^2 + m^2}{2m} = \frac{3m^2 - y^2}{2m} = HC$, p. Fig.
7	$\frac{9m^4 - 6m^2y^2 + y^4}{4m^2} + \frac{y^2}{4} = m^2$.
8	$5m^4 - 5m^2y^2 = -y^4$.
9	$m^4 - m^2y^2 = -2y^4$.
10	$m^4 - m^2y^2 + 25y^4 = 05y^4$.
11	$m^2 - y^2 = y^2 \sqrt{5}$.
12	$m^2 = y^2 + y^2 \sqrt{5} = y^2 \times \sqrt{5 + \sqrt{5}}$ = Sq. of (Radius
13	$m = y \times \sqrt{5 + \sqrt{5}}$
14	$2 = m \times \sqrt{2 + 2\sqrt{2}}$ = Diam. of Circle
15	$y \times \sqrt{5 - \sqrt{5}} = DX$ = side of Decagon

COROLLARY.

158. The Diameter of a Sphere, is in power to the Diameter of a Circle encompassing the Penta-

gon of a Dodecaedron, as $\frac{1}{1 + \sqrt{2}}$ to $\frac{3 - \sqrt{5}}{3}$.

H

For

$$\begin{array}{l}
 \text{For} \left\{ \begin{array}{l} 1 \left| 4R^2 \times \frac{3-\sqrt{5}}{6} = y^2 \right. \\ 2 \left| 4m^2 \times \frac{1}{2+2\sqrt{2}} = y^2. \right. \end{array} \right\} p. Art. \left\{ \begin{array}{l} 155. \\ 157. \end{array} \right. \\
 1=2 \quad 3 \left| 4R^2 \times \frac{3-\sqrt{5}}{6} = 4m^2 \times \frac{1}{2+2\sqrt{2}} \right. \\
 3 \text{ Analogy} \quad 4 \left| 4R^2 : 4m^2 :: \frac{1}{1+\sqrt{2}} : \frac{3-\sqrt{5}}{3} \right.
 \end{array}$$

S C H O L I U M.

159. According to this Method, I might proceed to investigate many Propositions belonging to this Body ; all of which would run into Surd Numbers : but my Design being Perspicuity, and Easiness of Expression, I have bethought me of another Method, wherein I have almost entirely avoided Surds ; which Method chiefly depends upon the following

L E M M A.

160. The Proportion which the side of a *Pentagon*, hath to the Diameter of its circumscribing Circle, is that of the Sine of 36 Degrees to the Radius.

Whoever does but a little consider the Nature of a *Pentagon*, will soon find that one half of its Side is the Sine of 36 Degrees, and the Distance from the Center to the middle of a Side is the Co-sine thereof, the Semidiameter of the circumscribing Circle being Radius. Then putting m for the said Semidiameter, and S for the Sine of 36 Degrees,

we

Algebraically Demonstrated.

51

we have by *Axiom I*, of *Plane Trigonometry*. As

$S: \frac{y}{2} :: R: m$, or as $y: 2m :: S: R$; putting R for the Sine of 90 Degrees.

P R O P. III.

161. The Diameter of a Sphere, is in power to the Diameter of a Circle encompassing the *Pentagon* of a *Dodecaedron*, as four times the Sine of 36 Degrees, is to $(b,)$ multiplied into the Square of the Radius, or Sine of 90 Degrees, or, (because $R = 1,00000$, &c.) as four times the Square of the said Sine to b .

$$\begin{array}{l} \text{For} \quad \left\{ \begin{array}{l} 1 \mid y = R\sqrt{b} \\ 2 \mid y = 2Sm \\ 3 \mid R\sqrt{b} = 2Sm. \\ 4 \mid 4R^2b = 16S^2m^2, \text{ i. e. } 4R^2 : 4m^2 :: 4S^2 : b. \end{array} \right\} \begin{array}{l} p. Art \left\{ \begin{array}{l} 155 \\ 160 \end{array} \right. \\ \\ \\ \end{array} \\ \begin{array}{l} I=2 \\ 3 \ominus 2 \& \times 4 \end{array} \end{array}$$

C O R O L L A R Y.

162. The Side of a *Dodecaedron*, is in power to the Diameter of the Circle encompassing a *Pentagonal* Face thereof, as S^2 is to 1.

$$\begin{array}{l} \text{For} \quad \left\{ \begin{array}{l} 1 \mid R^2b = y^2. \\ 2 \mid 4R^2 : y^2 :: 4 : b. \\ 3 \mid 4R^2 : 4m^2 :: 4S^2 : b. \\ 4 \mid y^2 : 4m^2 :: 4S^2 : 4 :: S^2 : 1. \end{array} \right. \begin{array}{l} 155 \\ \\ 161 \end{array} \\ \begin{array}{l} I \times 4 \& \text{Ano.} \\ \text{And} \\ \text{Therefore} \end{array} \end{array}$$

H 2

P R O P.

PROP. IV.

163. The Diameter of a *Sphere*, is in power to the Distance between the Center of the *Sphere*, or inscribed *Dodecaedron*, and middle of one of its *Pentagonal* Faces, or Height of one of the *Pyramids* which constitute the Solid, as four times the Square of the Sine of 36 Degrees, to the square of the said Sine made less by a fourth part of (*b*).

For if we imagine two Lines drawn from the Center of the *Dodecaedron*, one to the middle of one of the Faces, the other to an Angle of the said Face, we shall have a right angled Triangle, whose Hypothenufe is the Radius of the *Sphere*; the Perpendicular, the Radius of the Circle encompassing the *Pentagon*; and Base, the distance from the Center to the middle of the *Pentagon*, or the Height of one of the *Pyramids* which constitute the *Dodecaedron*, which Distance we call *n*.

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} n^2 = R^2 - m^2, \text{ per 47.1.} \\ m^2 = \frac{R^2 b}{4S^2} \cdot 161. \end{array} \right. \\
 \text{—} & & \\
 1, 2. & \left| \begin{array}{l} 3 \\ 4 \end{array} \right. & \left| \begin{array}{l} n^2 = R^2 - \frac{R^2 b}{4S^2} = \frac{R^2}{4S^2} \times 4S^2 - b = \\ \frac{4R^2}{16S^2} \times 4S^2 - b. \end{array} \right. \\
 3 \text{ Analogy} & \left| \begin{array}{l} 4 \end{array} \right. & \left| \begin{array}{l} 4R^2 : n^2 :: 16S^2 : 4S^2 - b :: 4S^2 : S^2 - \frac{1}{4}b. \end{array} \right.
 \end{array}$$

COROL.

COROLLARY.

164. The Diameter of a Circle encompassing the *Pentagon* of a *Dodecaedron*, is in power to the Height of the said *Pentagonal Pyramid*, as b to $S^2 - \frac{1}{4}b$.

For $\left\{ \begin{array}{l} 1 \left| \frac{1}{4}R^2 : 4m^2 :: 4S^2 : b \right. \\ 2 \left| \frac{1}{4}R^2 : n^2 :: 4S^2 : S^2 - \frac{1}{4}b \right. \\ 3 \left| 4m^2 : n^2 :: b : S^2 - \frac{1}{4}b \right. \end{array} \right\}$ per 161.
 But $\left. \begin{array}{l} \\ \\ \end{array} \right\}$ Art. 168.
 per 11.5.

LEMMA.

165. The Distance from the Center of a *Pentagon* to the middle of a Side, is, as the Sine of 54 Degrees, the Semidiameter of the Circle encompassing the *Pentagon* being Radius.

For all the Angles of a *Pentagon* are equal to six right Angles, per *Theo.* 1. 32, 1. each Angle therefore is equal to one fifth part of six right ones; i. e. equal to 108 Degrees; but this Line subtends only one half thereof, i. e. 54 Degrees, this Sine we will call (s .)

PROP. V.

166. The Expression for the Superficies of a *Dodecaedron* in the Terms of the Side, is $\frac{30y^2s}{2S} = \frac{15y^2s}{S}$.

But

	1	$\frac{1}{2} NPB = \text{Area of a } \textit{Pentagonal Face},$	(Art. 33.
But in this	2	$N=5, P=\frac{ys}{2S}, \text{ and } B=y$	
1, 2	3	$\frac{5y^2s}{2S} = \text{Area of one Face.}$	
3×12	4	$\frac{30y^2s}{2S} = \frac{15y^2s}{S} \text{ Superfi. of the } \textit{Dodecaedron}.$	

T H E O R E M.

Multiply 15 times the Sine of 54 Degrees, into the Square of the Side; and divide that Product by the Sine of 36 Degrees, that Quotient will be the Superficies.

L E M M A.

Fig. 16. 167. The natural Sine of 54 Degrees, multiplied into the Square Root of 3 times (ϕ) is equal to an Unit, or 1, *i. e.* $S\sqrt{\phi 3} = 1$.

For

For	{	1	$\frac{3m^2 - y^2}{2m} = \text{HC, per Step 6. Art. 157.}$	
		2	$3m^2 = y^2 \times \frac{15 + 3\sqrt{.05}}{1 + 2\sqrt{.05}}$	} <i>p. Ar. 157</i>
		3	$2m = 2y \times \sqrt{\frac{.5 + \sqrt{.05}}{.5 + \sqrt{.05}}}$	
2, 3		4	$\frac{3m^2 - y^2}{2m} = y \times \frac{.5 + 3\sqrt{.05}}{2\sqrt{.5 + \sqrt{.05}}} \quad (\text{HC})$	
Plane $\Delta^{\text{les.}}$		5	$S : R (=1) :: \frac{.5 + 3\sqrt{.05}}{2\sqrt{.5 + \sqrt{.05}}} (\text{NC}) :$ $\sqrt{.5 + \sqrt{.05}} (\text{CD}).$	
5 red		6	$S = \frac{.5 + 3\sqrt{.05}}{1 + 2\sqrt{.05}}$	
		7	$\sqrt{3b} = \sqrt{6 - 20\sqrt{.05}} \quad \text{per Art. 155.}$	
6 x 7		8	$S\sqrt{3b} = \frac{.5 + 3\sqrt{.05} \times \sqrt{6 - 20\sqrt{.05}}}{1 + 2\sqrt{.05}}$	
8 Abrev.		9	$S\sqrt{3b} = \frac{\sqrt{1.2 + \sqrt{.05}}}{1 + 2\sqrt{.05}} = \frac{1.2 + \sqrt{.8}}{1.2 + \sqrt{.8}} = 1.$	

168. The Expression for the Superficies of a *Dodecaedron* in the Terms of the circumscribing Sphere, is $\frac{5R^2}{sS}$.

For	{	1	$y^2 = R^2 b. \text{ per 155.}$
		2	$\frac{15sy^2}{S} = \text{Superficies. } p. 166.$
1, 2		3	$\frac{15sbR^2}{S} = \frac{5R^2}{sS} = \text{Superficies, 167.}$

T H E O R E M.

Divide five times the Square of the Radius of a *Sphere*, by the Rectangle of the Sines of 54 Degrees and 36 Degrees, and the Quotient will be the Superficies of the inscribed *Dodecaedron*.

P R O P. VI.

168. The Expression for the Solidity of a *Dodecaedron* in the Terms of the Side is $\frac{5sy^3}{2S^2} \sqrt{\frac{4S^2}{b} - 1}$

For	{	1	$\frac{15y^2s}{S} = \text{Superficies, per Art. 166.}$
		2	$n = R \times \frac{\sqrt{4S^2 - b}}{2S} \text{ per Art. 163.}$
$2 \div 3$		3	$\frac{n}{3} = R \times \frac{\sqrt{4S^2 - b}}{6S}$
		4	$R = \frac{y}{\sqrt{b}}, \text{ per Art. 155}$
$3, 4$		5	$\frac{y}{\sqrt{b}} \times \frac{\sqrt{4S^2 - b}}{6S} = \frac{y}{6S} \times \frac{\sqrt{4S^2 - b}}{b} - 1 = \frac{n}{3}$
1×5		6	$\frac{5sy^3}{2S^2} \sqrt{\frac{4S^2}{b} - 1} = \text{Solidity, p. Art. 98}$

T H E O R E M.

Divide four times the Square of the Sine of 36 Degrees, by b , and from that Quotient subtract 1; extract the Square Root of that Remainder, which Root, multiply into five times the Sine of 54 Degrees, and that Product into the Cube of the Side:

Side : Divide the last Product by twice the Square of the Sine of 36 Degrees, and the Quotient will be the Solidity.

169. The Expression for the Solidity in the Terms of the Diameter of the circumscribing *Sphere*,

is $\frac{5 s b R^3}{2 S^2} \sqrt{4 S^2 - b}$.

For
$$1 \times \frac{5 s}{2 S^2} \sqrt{\frac{4 S^2 - b}{b}} \left| \begin{array}{c} 1 \\ 2 \end{array} \right| \begin{array}{c} y^3 = R^3 b \sqrt{b}. \\ \frac{5 s b R^3}{2 S^2} \sqrt{4 S^2 - b} \end{array} \begin{array}{c} 155. \\ = \text{Solid.} \end{array}$$

THEOREM.

From four times the Square of the Sine of 36 Degrees, subtract b ; extract the Square Root of the Remainder; then multiply that Root, five times the Sine of 54 Degrees, b , and the Cube of the Radius of the Sphere, continually: Divide the last Product by 2 times the Square of the Sine of 36 Degrees, and the Quotient will be the Solidity.

P R O P. VII.

170. The Diameter of a *Sphere* circumscribing a *Dodecaedron*, is in power to the Diameter of a *Sphere* whose Superficies is equal to the Superficies of the *Dodecaedron*, as 4 $e s$ 5 is to 5.

I

For

For	{	1		$\frac{5R^2}{sS}$	}	= Superf. of the	{	<i>Dodecaed.</i> 168.
		2		$4r^2e$				<i>Sphere, &c.</i> 103.
1 = 2		3		$\frac{5R^2}{sS} = 4r^2e$, per Hypoth.				
3 redu.		4		$5R^2 = 4r^2e sS$.				
4 $\times \frac{1}{4}$		5		$20R^2 = 16r^2e sS$.				
5 Analogy		6		$4R^2 : 4r^2 :: 4e sS : 5$.				

P R O P. VIII.

171. The Diameter of a *Sphere* circumscribing a *Dodecaedron*, is in power to the Diameter of a *Sphere* whose Solidity is equal to the Solidity of the *Dodecaedron*, as $8eS$ is to $15sb\sqrt{4S^2-b}$.

For	{	1		$\frac{5sbR^3}{2S^2}\sqrt{4S^2-b}$	}	Solidity of the	{	<i>Dodecaed</i> (169.)
		2		$4x^3e$				<i>Sphere &c</i> (101.)
1 = 2		3		$\frac{5sbR^3}{2S^2}\sqrt{4S^2-b} = \frac{4x^3e}{3}$ per Hypoth.				
3 redu.		4		$15sbR^3\sqrt{4S^2-b} = 8x^3eS$.				
4 $\times 3$		5		$120sbR^3\sqrt{4S^2-b} = 64x^3eS$.				
5 Analog.		6		$8R^3 : 8x^3 :: 8eS : 15sb\sqrt{4S^2-b}$.				

C O R O L L A R Y.

172. The Diameter of the *Sphere* whose Superficies is equal to the Superficies of a *Dodecaedron*, is in power to the Diameter of a *Sphere* whose Solidity is equal to the Solidity of the *Dodecaedron*, as $R\sqrt{4S^2-b}$ to $2sx$.

For

For	{	1	$4R^2 : 4r^2 :: 4esS : 5 :: 8eS : \frac{10}{s}$	p. 170.
		2	$8R^3 : 8x^3 :: 8eS : 15sb\sqrt{4S^2-b}$	p. 171
$2 \div d$		3	$4R^2 : \frac{4x^2}{2R} :: 8eS : \frac{15sb}{2x}\sqrt{4S^2-b}$	
But		4	$\frac{4x^2}{2R} : 4r^2 :: \frac{15sb}{2x}\sqrt{4S^2-b} : \frac{10}{s}$	
4 red.		5	$4x^2 : 4r^2 :: 3Rs^2b\sqrt{4S^2-b} : 2x :: R\sqrt{4S^2-b} : 2sx$	

Section V. Of the ICOSAEDRON.

DEFINITION.

THE *Icosaedron* consisteth of twenty equal, Fig. 16.
and Equilateral Triangles; and is composed N. 1.
of twenty *Triangular Pyramids*, all equal one to
another, whose Vertex is the Center of the circum-
scribing *Sphere*.

Generation. The Generation of an *Icosaedron* may be thus conceived. Let there be a *Cylinder*, whose Altitude *FA* is equal to the Semidiameter of the Base *AO*, and suppose the regular *Pentagon* *ABCDE*, to be inscribed in it. Draw *BF*, the side of a *Decagon* and let fall the Perpendicular *Fa*, and (beginning at the point *A*) inscribe another *Pentagon* equal to the former in the lower Base of the *Cylinder*; then join *Aa*, *Bb*, *Ae*, *Be*, &c. all round the *Cylinder*; and it is evident that these Lines will form ten Triangles, which are equilateral, and have all the sides of each of them, equal to the side *AB*, of the *Pentagon*. Now if

we imagine the Lines OA, OB, OC, OD, OE , to be all raised up and extended, so that the point O is perpendicular over the Center, and each of them equal to the side AB of the *Pentagon*. And the same is conceived to be done in the lower Base of the *Cylinder*; we shall have five more equilateral Triangles above and five below, equal to the ten former ones; and so a Body, or space contained under twenty equal, and equilateral Triangles, will be generated, which is called an *Icosaedron*.

L E M M A.

173. The side of any regular *Pentagon*, *Decagon*, &c. is to the Diameter of its circumscribing Circle, as the side of any other regular *Pentagon*, *Decagon*, &c. is to the Diameter of its circumscribing Circle.

This is abundantly evident from *Plane Geometry*.

P R O P. I.

174. The Diameter of a *Sphere*, is in Power to the side of its inscribed *Icosaedron*, as $5 \times .5 \sqrt{5} + 1$ to 1.

Fig. 16.
N. 1.

For GO, Ho , are each of them the side of a *Decagon*, inscribed in the equal Circles $AEDCB, ABa$, and consequently equal to each other; and Oo the intermediate Part, is equal to the Radius of the Circle encompassing the *Pentagon* of the *Icosaedron*: But $GO + Ho (= 2 GO) + Oo$, = Diameter of the *Sphere*, per Figure; therefore putting $2R$ for the Diameter of the *Sphere*, and w for the side of the *Icosaedron*, we shall

Have

Have $\left\{ \begin{array}{l} 1 \quad 2w\sqrt{,5-\sqrt{,05}} = 2GO \\ 2 \quad w\sqrt{,5+\sqrt{,05}} = Oo \end{array} \right\} \text{ per Art. 157}$

$1+2 \quad 3 \quad 2w^2\sqrt{,5-\sqrt{,05}} + \sqrt{,5+\sqrt{,05}} = 2R.$

$3 \ominus 2 \quad 4 \quad 2w^2 - 4w^2\sqrt{,05} + 5w^2 + w^2\sqrt{,05} + 4w^2\sqrt{,25-\sqrt{,05}} = 4R^2.$

$4 \text{ contract.} \quad 5 \quad 2,5w^2 + 5w^2\sqrt{,05} = 5w^2 \times ,5 + \sqrt{,05} = 4R^2.$

$5 \text{ Analog.} \quad 6 \quad 4R^2 : w^2 :: 5 \times ,5 + \sqrt{,05} : 1.$

COROLLARIES.

175. The Diameter of a *Sphere* is in power to the Semidiameter of a Circle encompassing the *Pentagon* of an *Icosaedron* inscribed in the said *Sphere*, as 5 to 1.

(Circle. p. 157.

For $\left\{ \begin{array}{l} 1 \quad ,5w^2 + w^2\sqrt{,05} = \text{Sq. of the Semid. of the} \\ 2 \quad 5w^2 \times ,5 + \sqrt{,05} = \text{Sq. of Diam. Sphere p. 157.} \end{array} \right.$

But $\left\{ \begin{array}{l} 3 \quad 5w^2 \times ,5 + \sqrt{,05} : w^2 \times ,5 + \sqrt{,05} :: 5 : 1, \end{array} \right.$
(per 16.6.

176. The side of an *Icosaedron*, is in power to the Radius of the Circle encompassing the *Pentagon* of the *Icosaedron*, as 1 to $,5 + \sqrt{,05}$.

For $\left\{ \begin{array}{l} 1 \quad w^2 \\ 2 \quad w^2 \times ,5 + \sqrt{,05} \end{array} \right\} = \text{Sq. of } \left\{ \begin{array}{l} \text{Side p. 174.} \\ \text{Radius 157.} \end{array} \right.$

But $\left\{ \begin{array}{l} 3 \quad w^2 : w^2 \times ,5 + \sqrt{,05} :: 1 : ,5 + \sqrt{,05}. \end{array} \right.$

177. The Diameter of a *Sphere*, is in power to the Diameter of a Circle encompassing the *Triangle* of an *Icosaedron*, as 3 to $\frac{4}{5 \times ,5 + \sqrt{,05}}$.

For

For the Diameter of this Circle put $2p$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. & \left\{ \begin{array}{l} \frac{4R^2}{5+5+\sqrt{05}}=w^2 \\ 3p^2=w^2 \\ \frac{4R^2}{5 \times 5 + \sqrt{05}}=3p^2=\frac{12p^2}{4} \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 174. \\ 108. \end{array} \right. \\
 2 & & \\
 1=2 & & \\
 3 \text{ Anal} & \left\{ \begin{array}{l} 4 \end{array} \right. & \left\{ \begin{array}{l} 4R^2 : 4p^2 :: 3 : \frac{4}{5 \times 5 + \sqrt{05}} \end{array} \right.
 \end{array}$$

COROLLARIES.

178. The same Circle comprehends both the *Pentagon* of a *Dodecaedron*, and Triangle of an *Icosaedron* inscribed in the same *Sphere*.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. & \left\{ \begin{array}{l} \frac{3-\sqrt{5} \times 1+\sqrt{2}}{3} = \frac{2-4\sqrt{5}}{3} = 4m^2 \\ \frac{4}{3 \times 2,5 + 5\sqrt{5}} = 4p^2, p. \text{ Art. } 177. \\ 2-4\sqrt{5} = \frac{4}{25+5\sqrt{5}} \text{ per Hypoth.} \end{array} \right. \\
 1=2 & & \\
 3 \text{ redu.} & \left\{ \begin{array}{l} 4 \end{array} \right. & \left\{ \begin{array}{l} 4 = 4. \end{array} \right.
 \end{array}$$

179. The Diameter of a Circle encompassing the *Pentagon* of an *Icosaedron*, is in power to the Diameter of a Circle encompassing a Triangle of the same, as $6+6\sqrt{2}$ to 4.

Put M for the Diameter of the Circle encompassing the *Pentagon* of an *Icosaedron*.

For

$$\text{For } \left\{ \begin{array}{l} 1 \mid 4M^2 = w^2 \times 2 + 2\sqrt{2} \\ 2 \mid 4m^2 = \frac{4w^2}{3} \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 157. \\ 177. \end{array} \right.$$

$$\text{But } 3 \mid 4M^2 : 4m^2 :: 2 + 2\sqrt{2} : \frac{4}{3} :: 6 + 6\sqrt{2} : 4$$

N. B. I shall all along put $2m$ for the Diameter, both of the Circle encompassing the *Pentagon* of a *Dodecaedron*, and Triangle of the *Icosaedron* inscribed in the same *Sphere*.

P R O P. III.

180. The Diameter of a *Sphere*, is in power to the side of an *Icosaedron* inscribed in it, as 16 times the Square of the Sine of 36 Degrees, to $3b$.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \mid m^2 = \frac{R^2 b}{4S^2} \\ 2 \mid w^2 = 3m^2 \\ 3 \mid \frac{w^2}{3} = m^2 \\ 4 \mid \frac{R^2 b}{4S^2} = \frac{w^2}{3} \\ 5 \mid 12 R^2 b = 16 w^2 S^2. \\ 6 \mid 4 R^2 : w^2 :: 16 S^2 : 3b. \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 161. \\ 108. \end{array} \right. \\ 2 \div 3 \\ 1 = 3. \\ 4 \text{ red. } \& \times - \\ 5 \text{ Analogy} \end{array}$$

COROLLARY.

181. If a *Dodecaedron* and an *Icosaedron*, be both inscribed in the same *Sphere*; the Side of the *Dodecaedron*, will be in power to the Side of the *Icosaedron*, as 4 times the Square of the Sine of 36 Deg. to 3.

For

$$\begin{array}{l}
 \text{For } \left\{ \begin{array}{l} 1 \mid 4R^2 : 4 :: y^2 : b \\ 2 \mid 4R^2 : 16S^2 :: w^2 : 3b. \end{array} \right\} \text{ per } \left\{ \begin{array}{l} 155. \\ 180. \end{array} \right. \text{ Art.} \\
 1 \times 4 S^2 \mid 3 \mid 4R^2 : 16S^2 :: y^2 : 4bS^2. \\
 \text{But } 4 \mid y^2 : w^2 :: 4bS^2 : 3b :: 4S^2 : 3.
 \end{array}$$

P R O P. VI.

182. The Diameter of a *Sphere*, is in power to the perpendicular of a Triangle of an *Icosaedron* inscribed in it, as 64 times the Square of the Sine of 36 Degrees to $9b$. For this Perpendicular put (r .)

$$\begin{array}{l}
 \text{For } \left\{ \begin{array}{l} 1 \mid \frac{3m}{2} = r \\ 2 \mid m^2 = \frac{4r^2}{9} \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} \\ 108. \end{array} \right. \\
 1 \text{ redu.} \\
 \text{Also } 3 \mid m^2 = \frac{R^2 b}{4S^2} \quad 161. \\
 2 = 3 \quad 4 \mid \frac{R^2 b}{4S^2} = \frac{4r^2}{9}, \text{ i. e. } 4R^2 : r^2 :: 64S^2 : 9b.
 \end{array}$$

P R O P. V.

183. The Expression for the Superficies of an *Icosaedron*, in the Terms of the Side is, $5w^2\sqrt{3}$.

For $\left\{ \begin{array}{l} 1 \left| \frac{r}{2} = \frac{3R\sqrt{b}}{8S} \right. \\ 2 \left| w = \frac{R\sqrt{3b}}{2S} \right. \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 182. \\ 180. \end{array} \right.$

$1 \times 2 \quad 3 \quad \frac{3R^2b\sqrt{3}}{16S^2} = \text{Area of a triangular plain.}$

Also $4 \quad 3R^2b = 4S^2w^2.$

$3, 4 \quad 5 \quad \frac{w^2\sqrt{3}}{4} = \text{Superficies triangular plain.}$

$5 \times 20 \quad 6 \quad 5w^2\sqrt{3} = \text{Superficies, per Definition.}$

T H E O R E M.

Multiply 5 times the Square of the Side of an *Icofaedron*, into the Square Root of 3; that Product will be the Superficies.

184. The Expression for the Superficies of an *Icofaedron*, in the Terms of its circumscribing Sphere, is $\frac{15bR^2\sqrt{3}}{4S^2}.$

For $\left\{ \begin{array}{l} 1 \left| 5w^2\sqrt{3} = \text{Superficies per last.} \\ 2 \left| \frac{3bR^2}{4S^2} = w^2 \text{ p. 180.} \right. \\ 3 \left| \frac{15bR^2\sqrt{3}}{4S^2} = \text{Superficies per Subst.} \right. \end{array} \right.$

T H E O R E M.

Multiply 15 times the Square of the Radius of a *Sphere*, into b , and that Product into the Square Root of 3: Divide the last Product by 4 times the

K Square

Elements of SOLID GEOMETRY

Square of the Sine of 36 Degrees, the Quotient will be the Superficies of the inscribed *Icofaedron*.

185. N.B. The Height of a triangular *Pyramid* of the *Icofaedron*, is the same with the Height of a pentagonal *Pyramid* of the *Dodecaedron*: Or, the Distance from the middle, or Center of an *Icofaedron*, to the middle of a Triangular Face, is equal to the like Distance in the *Dodecaedron*. This is evident from *Art.* 178, and 143.

P R O P. VI.

186. The Expression for the Solidity of an *Icofaedron*, in the Terms of its Side is $\frac{5w^3}{3} \sqrt{\frac{4s^2-1}{b}}$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left\{ \begin{array}{l} 5w^2\sqrt{3} = \text{Superficies} \\ \frac{n}{3} = \frac{R}{6S} \times \sqrt{4s^2-1} \end{array} \right. \left. \begin{array}{l} \text{p. Art. } 183. \\ 163, 158 \end{array} \right. \\
 1 \times 2 & 3 & \frac{5w^2 R \sqrt{3}}{6S} \times \sqrt{4s^2-1} = \text{Sol. p. Art. } 98 \\
 \text{But} & 4 & R = \frac{2ws}{\sqrt{3b}} \text{ per Art. } 183 \\
 3, 4 & 5 & \frac{5w^3}{3} \times \sqrt{\frac{4s^2-1}{b}} = \text{Solidity.}
 \end{array}$$

T H E O R E M.

Divide 4 times the Square of the Sine of 36 Degrees by b ; from that Product subtract 1: Multiply the Square Root of the Remainder, into 5 times the Cube of the side, a third part of that Product will be the Solidity.

187. The

187. The Expression for the Solidity in the Terms of the circumscribing Sphere is $\frac{5bR^3}{8S^3} \sqrt{12S^2 - 3b}$.

For

$\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right.$	1	$\frac{5w^3}{3} \times \sqrt{\frac{4S^2}{b} - 1} = \text{Solidity } 186$
	2	$w = \frac{R\sqrt{3b}}{2S} \text{ per } 180.$
	3	$w^3 = \frac{3bR^3\sqrt{3b}}{8S^3}$
	4	$\frac{5bR^3}{8S^3} \times \sqrt{12S^2 - 3b} = \text{Solidity.}$

$2 \text{ \& } 3$
 $1, 3.$

THEOREM.

From 12 times the Square of the Sine of 36 Degrees, subtract $3b$; multiply the Square Root of the Remainder by 5 times b , and the Cube of the Radius of a Sphere, continually one into another; divide the last Product by 8 times the Cube of the Sine of 36 Degrees, and the Quotient will be the Solidity of the *Icosaedron* inscribed in that Sphere.

PROP. VI.

188. The Diameter of a Sphere circumscribing an *Icosaedron*, is in power to the Diameter of a Sphere, whose Superficies is equal to the Superficies of the *Icosaedron*, as $4S^2e$ is to $3\frac{1}{4}b\sqrt{3}$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \left| \frac{15bR^2\sqrt{3}}{4S^2} \right. \\ 2 \left| \frac{4r^2e}{4S^2} \right. \end{array} \right\} & = \text{Superfi.} \left\{ \begin{array}{l} \text{Icofaed. } 184 \\ \text{Sphere, \&c. } 103 \end{array} \right. \\
 1 = 2 & 3 & \frac{15bR^2\sqrt{3}}{4S^2} = 4r^2e, \text{ per Hypothefin.} \\
 3 \text{ redu.} & 4 & 15bR^2\sqrt{3} = 16S^2r^2e. \\
 4 \text{ Anal.} & 5 & 4R^2 : 4r^2 :: 4S^2e : 3\frac{1}{4}b\sqrt{3}.
 \end{array}$$

P R O P. VIII.

189. The Diameter of a *Sphere* circumscribing an *Icofaedron*, is in power to the Diameter of a *Sphere* whose Solidity is equal to the Solidity of the *Icofaedron*, as $4S^3e$ is to $\frac{15b}{8}\sqrt{12S^2-3b}$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \left| \frac{5bR^3}{8S^3}\sqrt{12S^2-3b} \right. \\ 2 \left| \frac{4x^3e}{3} \right. \end{array} \right\} & = \text{Sol.} \left\{ \begin{array}{l} \text{Icofaed. } 187 \\ \text{of the} \\ \text{Sphere, \&c. } 101 \end{array} \right. \\
 1 = 2 & 3 & \frac{5bR^3}{8S^3}\sqrt{12S^2-3b} = \frac{4x^3e}{3} \text{ per Hypoth.} \\
 3 \text{ redu.} & 4 & 15bR^3\sqrt{12S^2-3b} = 32S^3x^3e. \\
 4 \text{ Anal.} & 5 & 8R^3 : 8x^3 :: 4S^3e : \frac{15b}{8}\sqrt{12S^2-3b}
 \end{array}$$

C O R O L L A R Y.

190. The Diameter of a *Sphere* whose Solidity is equal to the Solidity of an *Icofaedron*, is in power to the Diameter of a *Sphere*, whose Superfi-

cies is equal to the Superficies of the Icofaedron, as
 $R\sqrt{4S^2 - b}$ is to $2xS$.

For	{	1		$4R^2 : 4r^2 :: 4S^2e : \frac{15b}{4}\sqrt{3} :: 4S^3e$ 188.
				$:\frac{15bS}{4}\sqrt{3}.$
		2		$8R^3 : 8x^3 :: 4S^2e : \frac{15b}{8}\sqrt{12S^2 - 3b}$ 189.
$2 \div d$		3		$4R^2 : \frac{4x^2}{2R} :: 4S^2e : \frac{15b}{16x}\sqrt{12S^2 - 3b}$
3, 1		4		$\frac{4x^2}{2R} : 4r^2 :: \frac{15b}{16x}\sqrt{12S^2 - 3b} :$ $\frac{15bS}{4}\sqrt{3}.$
4 red.		5		$4x^2 : 4r^2 :: R\sqrt{4S^2 - b} : 2xS.$

CHAP.

C H A P. II.

THAT we may demonstrate many delightful *Corollaries*, which naturally result from the foregoing *Propositions*, (when they are expressed in the Terms of some one Body ; but otherways very difficult, if not impossible to demonstrate ;) I have given their Expressions in the Terms of the *Cube*, finding it most proper for that Purpose ; because its Properties are expressed in simpler and easier Terms, than those of another Body.

Sect. I. *Of the SPHERE.*

P R O P. I.

191. The Expression for the Diameter of a *Sphere*, in the Terms of its inscribed *Cube*, is $x\sqrt{3}$.

$$\text{For } \begin{array}{c} 1 \\ 1 \end{array} \omega \frac{1}{2} \quad \left| \begin{array}{c} 1 \\ 2 \end{array} \right| \quad \begin{array}{l} 4 R^2 = 3x^2, \text{ per Art. 122.} \\ 2 R = x\sqrt{3}. \end{array}$$

T H E O R E M.

Multiply the side of a *Cube*, by the Square Root of 3 ; and the Product will be the Diameter of the circumscribing *Sphere*.

P R O P.

Algebraically Demonstrated.

PROP. I.

192. The Expression for the Superficies of a Sphere in the Terms of its inscribed Cube, is $3xe$.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} 4R^2 = 3x^2, \text{ per Art. 122.} \\ 4R^2e = \text{Superf. of the Sphere p. 103} \\ 4R^2e = 3x^2e = \text{Superficies.} \end{array}$$

THEOREM.

Multiply the Square of the side of a Cube, by 3 times the Ratio of the Diameter of a Circle to its Circumference, and the Product will be the Superficies of the circumscribing Sphere.

PROP. III.

193. The Expression for the Solidity of the said Sphere, is $\frac{x^3e\sqrt{3}}{2}$.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} R = \frac{x}{2} \sqrt{3}, \text{ per Art. 122.} \\ 3x^2e = \text{Superficies. per Art. 192.} \\ \frac{x^3e\sqrt{3}}{2} = \text{Solidity, per Art. 101.} \end{array}$$

THEOREM.

Multiply the Solidity of a Cube, the Ratio of the Diameter of a Circle to its Circumference, and the Square Root of 3, continually into one another; half the last Product will be the Solidity of its circumscribing Sphere.

Sect.

Sect. II. Of the TETRAEDRON.

P R O P. I.

194. The Expression for the side of a *Tetraedron*, in the Terms of a *Cube* inscribed in the same *Sphere*, is $x\sqrt{2}$.

$$\text{For } \left\{ \begin{array}{l|l} 1 & 4 R^2 = 3x^2 \\ 2 & 4 R^2 = \frac{3a^2}{2} \end{array} \right\} \text{ per } \left\{ \begin{array}{l} 122. \\ 109. \end{array} \right.$$

$$1 = 2 \quad \left\{ \begin{array}{l|l} 3 & 3x^2 = \frac{3a^2}{2} \end{array} \right\} \text{ per } 54.$$

$$3 \div 3 \&c. \quad \left\{ \begin{array}{l|l} 4 & x\sqrt{2} = a = \text{Side of the Tetraedron.} \end{array} \right.$$

T H E O R E M.

Multiply the side of a *Cube* by the Square Root of 2; the Product will be the side of a *Tetraedron* inscribed in the same *Sphere*.

P R O P. II.

195. The Expression for the Diameter of a Circle encompassing the Triangle of a *Tetraedron*, in the Terms of a *Cube*, &c. is $2x\sqrt{\frac{2}{3}}$.

$$\text{For } \left\{ \begin{array}{l|l} 1 & \frac{3^2 R^2}{9} = d^2 \\ 2 & R^2 = \frac{3x^2}{4} \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 110. \\ 122. \end{array} \right.$$

$$1, 2 \quad \left\{ \begin{array}{l|l} 3 & \frac{8x^2}{3} = d^2 \end{array} \right.$$

$$3 \text{ w } 2 \quad \left\{ \begin{array}{l|l} 4 & 2x\sqrt{\frac{2}{3}} = d = \text{Diameter, } \&c. \end{array} \right.$$

T H E O -

T H E O R E M.

Multiply the side of a *Cube*, by twice the Square Root of two thirds, and the Product will be the Diameter, &c.

P R O P. III.

196. The Expression for the Height *HI*, in the Terms of the *Cube* is $\frac{2x}{\sqrt{3}}$.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{a^2}{2R} = HI. \\ 2 \left| R = \frac{x}{2}\sqrt{3} \end{array} \right. \right\} \text{per Art.} \left\{ \begin{array}{l} 109. \\ 122. \end{array} \right. \\ 1, 2 \left| \begin{array}{l} 3 \left| \frac{a^2}{x\sqrt{3}} = HI. \text{ per Substitution.} \\ 4 \left| a^2 = 2x^2. \text{ Step 3, Art. 194.} \end{array} \right. \\ 3, 4 \left| \begin{array}{l} 5 \left| \frac{2x}{\sqrt{3}} = HI. \text{ per Substitution.} \end{array} \right. \end{array}$$

T H E O R E M.

Divide double the side of a *Cube* by the Square Root of 3, and the Quotient will be the Height of a *Tetraedron* inscribed in the same *Sphere*.

P R O P. IV.

197. The Expression for the Superficies of a *Tetraedron*, in the Terms of a *Cube* inscribed in the same *Sphere* is $2x^2\sqrt{3}$.

L

For

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left| \begin{array}{l} a^2 \sqrt{3} = \text{Superf. of the Tetraed. 114} \\ a^2 = 2x^2 \text{ p. Step 3. Art. 194.} \\ 2x^2 \sqrt{3} = \text{Superficies.} \end{array} \right.$$

THEOREM.

Multiply the Square of the side of a *Cube*, by twice the Square Root of 3, and the Product will be the Superficies of the *Tetraedron*, &c.

PROP. V.

198. The Expression for the Solidity of a *Tetraedron*, in the Terms of a *Cube* inscribed in the same *Sphere*, is $\frac{x^3}{3}$.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right\} \left| \begin{array}{l} \frac{a^3}{6\sqrt{2}} = \text{Solidity} \\ a^2 = 2x^2 \\ a = x\sqrt{2}. \\ a^3 = 2x^3\sqrt{2}. \\ \frac{2x^3\sqrt{2}}{6\sqrt{2}} = \frac{x^3}{3} = \text{Solidity.} \end{array} \right\} \begin{array}{l} \text{per} \\ \text{Art.} \end{array} \left\{ \begin{array}{l} 116. \\ 194. \end{array} \right.$$

THEOREM.

Divide the Solidity of a *Cube* by 3, and the Quotient will be the Solidity of a *Tetraedron*, &c.

Sect.

Sect. III. Of the OCTAEDRON.

PROP. I.

199. The Expression for the side of an *Octaedron*, in the Terms of a *Cube* inscribed in the same *Sphere*, is $x\sqrt{\frac{3}{2}}$.

For $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left\{ \begin{array}{l} 4R^2 = 3x^2 \\ 4R^2 = 2Z^2 \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 122. \\ 135. \end{array} \right.$

$1=2$ $\left\{ \begin{array}{l} 3 \\ 4 \end{array} \right\} \left\{ \begin{array}{l} 3x^2 = 2Z^2, \text{ per } 54. \\ x\sqrt{\frac{3}{2}} = Z = \text{side of the Octaedron.} \end{array} \right.$

THEOREM.

Multiply the side of a *Cube* by the Square Root of $\frac{3}{2}$ ($= 1.5$.) and the Product will be the side, &c.

PROP. II.

200. The Expression for the Perpendicular of a triangular Plane of the *Octaedron*, in the Terms of a *Cube* inscribed in the same *Sphere*, is $\frac{3x}{2\sqrt{2}}$.

For $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left\{ \begin{array}{l} 4R^2 = 3x^2 \\ 4R^2 = \frac{8p^2}{3} \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 122. \\ 140. \end{array} \right.$

$1=2$ $\left\{ \begin{array}{l} 3 \\ 4 \end{array} \right\} \left\{ \begin{array}{l} 3x^2 = \frac{8p^2}{3}, \text{ per } 54. \\ \frac{3x}{2\sqrt{2}} = p = \text{Perpendicular.} \end{array} \right.$

T H E O R E M.

Divide three times the side of a *Cube*, by twice the Square Root of 2; the Quotient will be the Perpendicular, &c.

P R O P. III.

201. The Expression for the Circle encompassing the Triangle of an *Octaedron*, in the Terms of a *Cube* in the same *Sphere*, is $x\sqrt{2}$.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} 4R^2 = 3x^2. \text{ p. 122.} \\ 4R^2 = \frac{3f^2}{2}. \text{ 137.} \\ 3x^2 = \frac{3f^2}{2}, \text{ i. e. } x\sqrt{2} = f = \text{Dia-} \\ \text{meter, \&c. 54.} \end{array} \right.$$

P R O P. IV.

202. The Expression for the Superficies of an *Octaedron* in the Terms of the *Cube*, is $3x^2\sqrt{3}$.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} 2Z^2\sqrt{3} = \text{Superf.} \\ 2Z^2 = 3x^2 \\ 3x^2\sqrt{3} = \text{Superficies in the Terms} \\ \text{of the Cube.} \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 143. \\ 199. \end{array} \right.$$

T H E O R E M.

Multiply the Square of the Side of any *Cube*, by three times the Square Root of 3; and the Product will be the Superficies of an *Octaedron* inscribed in the same *Sphere*.

P R O P.

P R O P. V.

203. The Expression for the Solidity of an *Octaedron*, in the Terms of the *Cube*, is $\frac{x^3}{2} \sqrt{3}$.

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{Z^3 \sqrt{2}}{3} = \text{Solidity} \right. \\ 2 \left| Z = x \sqrt{\frac{3}{2}} \right. \\ 3 \left| Z^3 = \frac{3x^3}{2} \sqrt{\frac{3}{2}} \right. \\ 4 \left| \frac{x^3}{2} \sqrt{3} = \text{Solidity in the Terms of the Cube.} \right. \end{array} \right\} \text{per Art. } \left\{ \begin{array}{l} 148. \\ 199. \end{array} \right.$$

T H E O R E M.

Multiply half the Solidity of a *Cube*, by the Square Root of 3; and the Product will be the Solidity of an *Octaedron* inscribed in the same *Sphere*.

Sect. IV. *Of the DODECAEDRON.*

P R O P. I.

204. The Expression for the side of a *Dodecaedron* in the Terms of the *Cube*, is $x \times \sqrt{\frac{3 - \sqrt{5}}{2}}$ or

$$\frac{x}{2} S^{\circ}$$

For

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} & \left| \begin{array}{l} 4 R^2 = 3x^2 \\ 4 R^2 = \frac{6y^2}{3-\sqrt{5}} \end{array} \right. & \left. \right\} p. Art. \left\{ \begin{array}{l} 122. \\ 155. \end{array} \right. \\
 1 = 2. & 3 & 3x^2 = \frac{6y^2}{3-\sqrt{5}} \text{ per } 54 \\
 3 \text{ red.} & 4 & x^2 \times \frac{3-\sqrt{5}}{6} = 2y^2. \\
 4 \text{ } w \div 2 & 5 & x \times \sqrt{\frac{3-\sqrt{5}}{2}} = y.
 \end{array}$$

T H E O R E M.

From 3 subtract the Square Root of 5, half the Square Root of the Remainder multiplied into the Side of a *Cube*, the Product will be the Side of a *Dodecaedron* inscribed in the same *Sphere*.

The last another way.

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} & \left| \begin{array}{l} 4 R^2 = 3x^2 \\ 4 R^2 = \frac{4y^2}{b} \end{array} \right. & \left. \right\} per Art. \left\{ \begin{array}{l} 122. \\ 155. \end{array} \right. \\
 1 = 2 & 3 & 3x^2 = \frac{4y^2}{b} \text{ per } 54. \\
 3 \text{ red. \& } w 2 & 4 & \frac{x}{2} \sqrt{3b} = \frac{x}{2} S \text{ (per Art. 167.) } = y.
 \end{array}$$

T H E O R E M.

Divide the Side of a *Cube* by twice the Sine of 54 Degrees, and the Quotient will be the Side of a *Dodecaedron* inscribed in the same *Sphere*.

P R O P.

P R O P. II.

205. The Expression for the Diameter of a Circle encompassing the *Pentagon* of a *Dodecaedron* in the Terms of the *Cube* is $\frac{x}{2sS}$.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \left| 4R^2 = 3x^2 \right. \\ 2 \left| 4R^2 = \frac{16m^2 S^2}{b} \right. \end{array} \right\} \text{ per } \left\{ \begin{array}{l} 122. \\ 161. \end{array} \right. \\ \\ 1 = 2 \quad 3 \left| 3x^2 = \frac{16m^2 S^2}{b} \right. \text{ per } 54 \\ \\ 3 \times 8 \div 4 \left| \begin{array}{l} 3bx^2 \\ 16S^2 = m^2. \end{array} \right. \\ \\ 4 \times 4 \times 2 \left| 5 \left| \frac{x\sqrt{3b}}{2S} = \frac{x}{2sS} \text{ (per Art. 167.)} = 2m \right. \right. \end{array}$$

T H E O R E M.

Divide the side of a *Cube* by twice the Rectangle of the Sine of 54 Degrees and Sine of 36 Degrees, and the Quotient will be the Diameter of a Circle, &c.

P R O P. III.

206. The Expression for the Distance from the Center of a *Dodecaedron*, to the Center of a Circle encompassing one of the *Pentagonal* Planes, or the Height of a *Pentagonal* Pyramid of the *Dodecaedron*, is

$$\frac{x}{2} \sqrt{3 - \frac{3b}{4S^2}}.$$

For

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left\{ \begin{array}{l} R^2 = \frac{3x^2}{4} \\ R^3 = \frac{n^2}{1 - \frac{b}{4S^2}} \end{array} \right. & \begin{array}{l} 122. \\ \text{per Ar.} \\ 163. \end{array} \\
 1 = 2 & 3 & \frac{3x^2}{4} = \frac{n^2}{1 - \frac{b}{4S^2}} \quad \text{per 54.} \\
 3 \text{ redu.} & 4 & x^2 \times 3 - \frac{3b}{4S^2} = 4n^2 \\
 4w2\&\div 2 & 5 & \frac{x}{2} \sqrt{3 - \frac{3b}{4S^2}} = n
 \end{array}$$

THEOREM.

Divide three times b , by four times the Square of the Sine of 36 Degrees, subtract that Quotient from 3; then the Square Root of the Remainder multiplied into half the side of a *Cube*, the Product will be the Height of a *Pyramid* of a *Dodecaedron*, &c.

PROP. III.

207. The Expression for the Superficies of a *Dodecaedron* in the Terms of the *Cube* is $\frac{15x^2}{4SS}$.

For

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left. \begin{array}{l} R^2 = \frac{3x^2}{4} \\ R^2 = \frac{y^2}{b} \end{array} \right\} \begin{array}{l} \text{per Art.} \\ \text{per Art.} \end{array} \begin{array}{l} 122. \\ 155. \end{array} \\
 1 = 2 & 3 & \frac{3x^2}{4} = \frac{y^2}{b} \text{ P. 54} \\
 3 \text{ red.} & 4 & \frac{3bx^2}{4} = y^2 \\
 \text{But} & 5 & \frac{15sy^2}{S} = \text{Superficies per Art. 166.} \\
 4, 5 & 6 & \frac{45sbx^2}{4S} = \frac{15x^2}{4sS} = \text{Superfi. 167}
 \end{array}$$

T H E O R E M.

Multiply the Square of the side of a *Cube* by 15; divide that Product by 4 times the Rectangle of the Sine of 36 Degrees, and Sine of 54 Degrees, the Quotient is the Superficies of a *Dodecaedron*, &c.

P R O P. V.

208. The Expression for the Solidity of a *Dodecaedron* in the Terms of the *Cube*, is $\frac{5x^3}{16sS^2} \times \sqrt{12S^2 - 3b}$.

M

For

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{45sbx^2}{4S} = \text{Superficies} \right. \\ 2 \left| \frac{x}{12S} \sqrt{12S^2 - 3b} = \frac{n}{3} \right. \\ 3 \left| \frac{5x^3}{16sS^2} \times \sqrt{12S^2 - 3b} = \text{Solidity } p. \text{ Art.} \right. \end{array} \right\} \begin{array}{l} \text{per} \\ \text{Art.} \end{array} \left\{ \begin{array}{l} 207 \\ 206 \end{array} \right.$$

1 x 2, &c.

THEOREM.

Subtract $3b$, from 12 times the Square of the Sine of 36 Degrees; multiply the Square Root of that Remainder, into 5 times the Solidity of a *Cube*; divide the Product by 16 times the Sine of 54 Degrees multiplied into the Square of the Sine of 36 Degrees, the Quotient will be the Solidity of an *Icosaedron*, &c.

Sect. V. Of the ICOSAEDRON.

209. The Expression for the side of an *Icosaedron* in the Terms of the *Cube* is $\frac{x\sqrt{3}}{\sqrt{25 \times \sqrt{.05}}} \text{ or } \frac{3x\sqrt{b}}{4S}$.

$$\text{For } \left\{ \begin{array}{l} 1 \left| 4R^2 = 3x^2 \\ 2 \left| 4R^2 = w^2 \times 2,5 + 5\sqrt{.05} \right. \\ 3 \left| 3x^2 = w^2 \times 2,5 + 5\sqrt{.05} \right. \\ 4 \left| \frac{3x^2}{2,5 + 5\sqrt{.05}} = w^2 \\ 5 \left| \frac{x\sqrt{3}}{\sqrt{2,5 + 5\sqrt{.05}}} = w. \right. \end{array} \right\} \begin{array}{l} \text{per} \\ \text{Art.} \end{array} \left\{ \begin{array}{l} 122. \\ 174. \end{array} \right.$$

1 = 2
3 ÷
4 w 2

The

The same another Way.

$$\text{For } \left\{ \begin{array}{l|l} 1 & 4R^2 = 3x^2 \\ 2 & 4R^2 = \frac{16w^2 S^2}{3b} \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 122. \\ 180. \end{array} \right.$$

$$1 = 2 \quad 3 \quad 3x^2 = \frac{16w^2 S^2}{3b}.$$

$$3 \text{ redu.} \quad 4 \quad 9bx^2 = 16w^2 S^2$$

$$4 \div 16 S^2 \quad 5 \quad \frac{3x\sqrt{b}}{4S} = w.$$

$$\text{and } w 2$$

THEOREM.

Multiply the side of a *Cube* by three times the Square Root of b ; and divide that Product by 4 times the Sine of 36 Degrees, the Quotient will be the side of an *Icosaedron* inscribed in the same *Sphere*.

PROP. II.

210. The Expression for the Perpendicular of a Triangle of the *Icosaedron* in the Terms of the *Cube*, is $\frac{3x}{8S}$.

M 2

For

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} R^2 = \frac{3x^2}{4} \\ R^2 = \frac{16S^2 r^2}{9b} \end{array} \right. & \left. \right\} \text{per Art. } \left\{ \begin{array}{l} 122. \\ 132. \end{array} \right. \\
 1=2 & 3 & \frac{3x^2}{4} = \frac{16S^2 r^2}{9b} \\
 3 \text{ red.} & 4 & \frac{27br^2}{64S^2} = r^2 \\
 4 \text{ w } 2 & 5 & \frac{3x\sqrt{3b}}{8S} = \frac{3x}{8S} \text{ (per Art. 167.)} = r
 \end{array}$$

T H E O R E M.

Divide three times the side of a *Cube*, by 8 Rectangles of the Sine of 36 Degrees, and the Sine of 54 Degrees; the Quotient will be the Perpendicular of a Triangle of the *Icofaedron*, &c.

P R O P. III.

211. The Expression for the Superficies of an *Icofaedron* in the Terms of a *Cube*, is $\frac{45bx^2\sqrt{3}}{16S^2}$.

For

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} R^2 = \frac{3x^2}{4} \text{ per Art. 122} \\ \frac{15bR^2\sqrt{3}}{4S^2} \\ \frac{45bx^2\sqrt{3}}{16S^2} \end{array} \right\} = \text{Superfi.} \left\{ \begin{array}{l} \text{Sphere 184} \\ \text{Cube} \end{array} \right.$
 in the terms of the

THEOREM.

Multiply 45, b , the Square of the side of a *Cube*, and the Square Root of 3 continually one into another; divide the last Product by 16 times the Square of the Sine of 36 Degrees, and the Quotient will be the Superficies of an *Icosaedron*, &c.

PROP. IV.

212. The Expression for the Solidity of an *Icosaedron* in the Terms of the *Cube*, is $\frac{45bx^3}{64S^3} \times \sqrt{4S^2 - b}$.

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right| \begin{array}{l} R^2 = \frac{3x^2}{4} \text{ per Art. 122} \\ R = \frac{x}{2}\sqrt{3} \\ R^3 = \frac{3x^3}{8}\sqrt{3} \\ \frac{5bR^3}{8S^3} \times \sqrt{12S^2 - 3b} = \text{Solidity 187} \\ \frac{15bx^3\sqrt{3}}{64S^3} \times \sqrt{12S^2 - 3b} = \frac{45bx^3}{64S^3} \times \sqrt{4S^2 - b} = \text{Solidity.} \end{array} \right.$

The

The last another Way.

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{45 b x^2 \sqrt{3}}{16 S^2} = \text{Superf. 211} \right. \\ 2 \left| \frac{x}{4 S} \times \sqrt{12 S^2 - 3b} = \text{Alt. } \Delta \text{lar Pyr. } p. \text{ Ar. 206} \right. \\ 3 \left| \frac{45 b x^3}{64 S^3} \times \sqrt{4 S^2 - b} = \text{Solidity.} \right. \end{array} \right.$$

$1 \times \frac{2}{3} \text{ of } 2$

THEOREM.

From the Square of twice the Sine of 36 Degrees subtract b , multiply the Square Root of the Remainder, the Solidity of a *Cube*, 45, and b , continually one into another, and divide the last Product by the *Cube* of 4 times the Sine of 36 Degrees, the Quotient shall be the Solidity of the *Icosaedron*, &c.

General COROLLARIES.

The *Corollaries* which naturally and easily result from the *Propositions* in this *Chapter*, are (both for Multitude and Worth) really wonderful! out of which Variety I have selected these few which follow, leaving the rest for the Learner to do at his Pleasure.

213. 1. The Solidity of a *Tetraedron*, is to the Solidity of a *Cube* inscribed in the same *Sphere*, as 1 to 3.

For

For $\left\{ \frac{x^3}{3} \right\} = \text{Solidity of the } \left\{ \begin{array}{l} \text{Tetraed.} \\ \text{Cube} \end{array} \right\} \text{p. Art. } \left\{ \begin{array}{l} 198 \\ 128 \end{array} \right.$

But $\frac{x^3}{3} : x^3 :: 1 : 3$ per 16. 6.

214. 2. The Solidity of an *Octaedron*, is to treble the Solidity of a *Tetraedron* inscribed in the same *Sphere*, as their sides are.

For $\left\{ \begin{array}{l} x^3 \sqrt{\frac{3}{4}} \\ x^3 \\ x\sqrt{\frac{3}{2}} \\ x\sqrt{2} \end{array} \right\} = \text{to the } \left\{ \begin{array}{l} \text{Solidity} \\ \text{treble Sol.} \\ \text{side of the} \end{array} \right\} \left\{ \begin{array}{l} \text{Octaed. 203} \\ \text{Tetraed. 198} \\ \text{Octaed. 199} \\ \text{Tetraed. 194} \end{array} \right.$

But $x^3 \sqrt{\frac{3}{4}} : x^3 :: x\sqrt{\frac{3}{2}} : x\sqrt{2}$, per 16.6

215. 3. The Superficies of a *Cube*, is to the Superficies of an *Octaedron* inscribed in the same *Sphere*, as their Solidities.

For $\left\{ \begin{array}{l} 6x^2 \\ 3x^2\sqrt{3} \\ x^3 \\ \frac{x^3\sqrt{3}}{2} \end{array} \right\} = \text{to the } \left\{ \begin{array}{l} \text{Superfi.} \\ \text{of the} \\ \text{Solidity} \\ \text{of the} \end{array} \right\} \left\{ \begin{array}{l} \text{Cube} \\ \text{Octaed.} \\ \text{Cube} \\ \text{Octaed.} \end{array} \right\} \left\{ \begin{array}{l} \text{per} \\ \text{Art.} \\ \text{per} \\ \text{Art.} \end{array} \right\} \left\{ \begin{array}{l} 126 \\ 202 \\ 128 \\ 203 \end{array} \right.$

But $6x^2 : 3x^2\sqrt{3} :: x^3 : \frac{x^3\sqrt{3}}{2}$ per 16.6.

216. 4. The side of a *Cube* is in power to the Diameter of a Circle encompassing the Triangle of a *Tetraedron* inscribed in the same *Sphere*, as 3 to 8.

For

$$\text{For } \left\{ \frac{x^2}{3} \right\} = \text{Square of the } \left\{ \begin{array}{l} \text{Side of Cube} \\ \text{Diam. of the Circle} \end{array} \right\} \text{ per } \left\{ \begin{array}{l} 122 \\ 195 \end{array} \right.$$

$$\text{But } x^2 : \frac{8x^2}{3} :: 3 : 8, \text{ per Art. 16.6.}$$

217. 5. The Solidity of a *Cube*, is to the Solidity of an *Octaedron* inscribed in the same *Sphere*, as the side of the *Cube*, is to the Radius of the *Sphere*.

$$\text{For } \left\{ \begin{array}{l} x^3 \\ x^3 \sqrt{\frac{1}{4}} \\ x \\ x \sqrt{\frac{1}{4}} \end{array} \right\} = \text{to the } \left\{ \begin{array}{l} \text{Solidity of the } \left\{ \begin{array}{l} \text{Cube} \\ \text{Octaed.} \end{array} \right\} \text{ per } \left\{ \begin{array}{l} 128 \\ 203 \end{array} \right\} \\ \text{Side of the Cube} \\ \text{Rad. of Sphere} \end{array} \right\} \text{ per } \left\{ \begin{array}{l} 122 \\ 191 \end{array} \right.$$

$$\text{But } x^3 : x^3 \sqrt{\frac{1}{4}} :: x : x \sqrt{\frac{1}{4}}, \text{ per 16. 6.}$$

217. 6. The side of a *Tetraedron*, is to the side of an *Octaedron*, as the Height of the *Tetraedron*, is to the side of a *Cube*, all being inscribed in the same *Sphere*.

$$\text{For } \left\{ \begin{array}{l} x \sqrt{2} \\ x \sqrt{\frac{1}{2}} \\ \frac{2x}{\sqrt{3}} \\ x \end{array} \right\} = \text{to the } \left\{ \begin{array}{l} \text{Side of a } \left\{ \begin{array}{l} \text{Tetraed.} \\ \text{Octaed.} \end{array} \right\} \\ \text{Height of a } \left\{ \begin{array}{l} \text{Tetraedron} \\ \text{Side of Cube.} \end{array} \right\} \end{array} \right\} \text{ p. Art } \left\{ \begin{array}{l} 194 \\ 199 \\ 196 \\ 122 \end{array} \right.$$

$$\text{But } x \sqrt{2} : x \sqrt{\frac{1}{2}} :: \frac{2x}{\sqrt{3}} : x, \text{ p. 16.6.}$$

218. 7. If

218. 7. If a right Line be divided in mean and
extream Proportion, the lesser Segment shall be the
side of a *Dodecaedron*, and the greater Segment shall
be the side of a *Cube* inscribed in the same *Sphere*.

For { 1 | $x \times \sqrt{\frac{3-\sqrt{5}}{2}} : x :: x : x + x \times$
 $\frac{\sqrt{3-\sqrt{5}}}{2}, \text{ per Hypoth.}$

1 ∴ 2 | $x^2 \times \sqrt{\frac{3-\sqrt{5}}{2}} + x^2 \times \frac{3-\sqrt{5}}{2} = x^2$

2 ÷ x^2 3 | $\sqrt{\frac{3-\sqrt{5}}{2}} + \frac{3-\sqrt{5}}{2} = 1$

3 × 2 4 | $\sqrt{6-2\sqrt{5}} + 3-\sqrt{5} = 2.$

4-3+ $\sqrt{5}$ 5 | $\sqrt{6-2\sqrt{5}} = \sqrt{5}-1.$

5 ⊙ 2 6 | $6-2\sqrt{5} = 6-2\sqrt{5}, \text{ therefore, \&c.}$

219. 8. The Superficies of a *Dodecaedron*, is to
the Superficies of an *Icosaedron*, as the side of a
Cube is to the side of the *Icosaedron*; all being in-
scribed in the same *Sphere*.

For { 1 | $\frac{15x^2}{4S} : \frac{45bx^2\sqrt{3}}{16S^2} :: x : \frac{3x\sqrt{b}}{4S} \left\{ \begin{array}{l} \text{per} \\ \text{Hyp.} \end{array} \right.$

1 ∴ 2 | $\frac{45x^3\sqrt{b}}{16S^2} = \frac{45bx^3\sqrt{3}}{16S^2}$

But 3 | $\frac{\sqrt{b}}{S} = b\sqrt{3}, \text{ per Art. 167.}$

2, 3 4 | $\frac{45bx^3\sqrt{3}}{16S^2} = \frac{45bx^3\sqrt{3}}{16S^2} \left\{ \begin{array}{l} \text{From the} \\ \text{Equality of the Terms the} \\ \text{Prop. is evident.} \end{array} \right.$

N

220. 9. The

220. 9. The Solidity of a *Dodecaedron*, is to the Solidity of an *Icosaedron*, as the side of a *Cube*, is to the side of the *Icosaedron*; all being inscribed in the same *Sphere*.

For	1	$\frac{5x^3}{16sS^2} \times \sqrt{12S^2 - 3b} = \frac{5x^3\sqrt{3}}{16sS^2} \times$ $\sqrt{4S^2 - b} = \text{Sol. Dodecaed. } 208$
	2	$\frac{45bx^3}{64S^3} \times \sqrt{4S^2 - b} = \text{Sol. Icosaed. } 212$
	3	$\frac{3x\sqrt{b}}{4S} = \text{side of the Icosaed. } 209$
Then	4	$\frac{5x^3\sqrt{3}}{16sS^2} \times \sqrt{4S^2 - b} : \frac{45bx^3}{64S^3} \times$ $\sqrt{4S^2 - b} :: x : \frac{3x\sqrt{b}}{4S} \text{ per Hyp.}$
4 ∴	5	$\frac{15x^4\sqrt{3b}}{64sS^2} \times \sqrt{4S^2 - b} = \frac{45bx^4}{64S^3} \times$ $\sqrt{4S^2 - b}.$
But	6	$\frac{\sqrt{3b}}{s} = 3b, \text{ p. Art. 167}$
5, 6	7	$\frac{15x^4}{64S^3} \times \sqrt{4S^2 - b} = \frac{15x^4}{64S^3} \times \sqrt{4S^2 - b}$
		Therefore the Proportion is evident, per 16. 6.

221. 10. From the two last *Corollaries* it is evident, that the Superficies of a *Dodecaedron*, is to the Superficies of an *Icosaedron* inscribed in the same *Sphere*, as their Solidities are.

CHAP.

C H A P. III.

Of the SPHERE, inscribed within any of these Bodies.

222. **F**OR finding the Diameter of the *Sphere* inscribed within any of these Bodies, take this General Rule.

R U L E.

From the Square of the Radius of the circumscribing *Sphere*, subtract the Square of the Radius of the Circle encompassing,

The $\left\{ \begin{array}{l} \text{Triangle of a } \left\{ \begin{array}{l} \text{Tetraedron,} \\ \text{Octaedron,} \\ \text{Icosaedron,} \end{array} \right. \text{ And the Sq.} \\ \text{Square of a Hexagon,} \\ \text{Pentagon of a Dodecaedron,} \end{array} \right. \left. \begin{array}{l} \text{Root of the Re-} \\ \text{mainder, is the} \\ \text{Radius of the in-} \\ \text{scribed Sphere.} \end{array} \right.$

For, whosoever does but consider these *Bodies*, will see, that the Radius of the inscribed *Sphere*, is the Base of a Right-angled Triangle, whose Hypothenuse is the Radius of the circumscribing *Sphere*, and the Perpendicular the Radius of the Circle encompassing one of the Planes which terminates the Body.

Sect. I. Of the TETRAEDRON.

P R O P. I.

223. The Expression for the Diameter of a *Sphere* inscribed within a *Tetraedron*, in the Terms of its circumscribing *Sphere*, is $\frac{2 R}{3}$.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right\} \left| \begin{array}{l} R^2 = \\ \frac{8 R^2}{9} = \\ R^2 \\ \frac{2 R}{3} \end{array} \right. \left. \begin{array}{l} \text{Square of} \\ \text{Rad. of the} \\ \text{circumscrib.} \end{array} \right\} \begin{array}{l} \text{Sphere } p. 109 \\ \text{Circle } 110 \end{array} \\ \begin{array}{l} 1 - 2 \\ 3 \times 2 \end{array} \left| \begin{array}{l} \frac{R^2}{9} = \text{Sq. Rad. of inscrib. Sphere, per} \\ \frac{2 R}{3} = \text{Diam. of the inscribed Sphere,} \end{array} \right. \begin{array}{l} \text{(Gen. Rule)} \\ \end{array} \end{array}$$

That is, the Diameter of a *Sphere* circumscribing a *Tetraedron*, is in power to the Diameter of a *Sphere* inscribed within it, as 9 to 1.

COROLLARIES.

224. The side of a *Tetraedron*, is in power to the Diameter of a *Sphere* inscribed within it, as 6 to 1.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left| \begin{array}{l} 4 R^2 : a^2 :: 3 : 2 :: 9 : 6 \\ 4 R^2 : \frac{4 R^2}{9} :: 9 : 1 \end{array} \right. \left. \begin{array}{l} \text{per } \left\{ \begin{array}{l} 109 \\ 223 \end{array} \right\} \\ \text{Ar. } \left\{ \begin{array}{l} 109 \\ 223 \end{array} \right\} \end{array} \right. \\ \text{That is, } \left| \begin{array}{l} 3 \\ 4 \end{array} \right| \left| \begin{array}{l} a^2 : \frac{4 R^2}{9} :: 6 : 1, \text{ per } 11.5. \end{array} \right. \end{array}$$

225. The

225. The Height of a *Tetraedron*, is in power to the Diameter of a *Sphere* inscribed within it, as 4 to 1.

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} 4R^2 : b^2 :: 9 : 4 \\ 4R^2 : \frac{4R^2}{9} :: 9 : 1 \\ b^2 : \frac{4R^2}{9} :: 4 : 1, \text{ per 11. 5.} \end{array} \right\} p. Art. \left\{ \begin{array}{l} 111 \\ 223 \end{array} \right.$

That is,

P R O P. II.

226. The side of a *Tetraedron* inscribed within the inscribed *Sphere*, (or second *Tetraedron*) is in power to the Diameter of the first circumscribing *Sphere*, as 2 to 27.

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} \frac{4R^2}{9} = \text{Square of the Diam. of the} \\ \text{inscribed Sphere, per Art. 223} \\ 3 : 2 :: \frac{4R^2}{9} : \frac{8R^2}{27} = \text{Square of} \\ \text{(the side, per Art. 109.} \\ \frac{8R^2}{27} : 4R^2 :: 2 : 27, \text{ per 16.6.} \end{array} \right.$

But

That is

N. B. The Bodies inscribed within the first circumscribing *Sphere*, I shall (for Distinction Sake) call First; and those inscribed within the inscribed *Sphere*, I shall call Second Bodies.

COROL-

COROLLARIES.

227. The side of the first *Tetraedron*, is in power to the side of the second *Tetraedron*, as 9 to 1.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{8R^2}{3} = \right. \\ 2 \left| \frac{8R^2}{27} = \right. \end{array} \right. \left. \begin{array}{l} \text{Sq. of the} \\ \text{side of the} \end{array} \right\} \left\{ \begin{array}{l} \text{1st} \\ \text{2d} \end{array} \right\} \left. \begin{array}{l} \text{Tetraed.} \\ \text{per Art.} \end{array} \right\} \begin{array}{l} 109 \\ 226 \end{array} \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{8R^2}{3} : \frac{8R^2}{27} :: 27 : 3 :: 9 : 1 \text{ per 16.6} \right. \end{array} \right\}$$

228. The Diameter of the first circumscribing *Sphere*, is to the Diameter of the inscribed *Sphere*; as the side of the first *Tetraedron*, is to the side of the second *Tetraedron*.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{2R}{2} = \right. \\ 2 \left| \frac{2R}{3} = \right. \end{array} \right\} \left. \begin{array}{l} \text{Diameter of the} \\ \text{circum. Sphere} \\ \text{inscrib. Sphere} \end{array} \right\} \left\{ \begin{array}{l} \text{per Art.} \end{array} \right\} \begin{array}{l} 109 \\ 223 \end{array} \\ \left\{ \begin{array}{l} 3 \left| \frac{2R\sqrt{2}}{\sqrt{3}} = \right. \\ 4 \left| \frac{2R\sqrt{2}}{3\sqrt{3}} = \right. \end{array} \right\} \left. \begin{array}{l} \text{side} \\ \text{of the} \end{array} \right\} \left\{ \begin{array}{l} \text{1st} \\ \text{2d} \end{array} \right\} \left. \begin{array}{l} \text{Tetraed.} \\ \text{per Art.} \end{array} \right\} \begin{array}{l} 109 \\ 226 \end{array} \\ \text{But} \left\{ \begin{array}{l} 5 \left| 2R : \frac{2R}{3} :: \frac{2R\sqrt{2}}{\sqrt{3}} : \frac{2R\sqrt{2}}{3\sqrt{3}}, \text{ per 16.6} \right. \end{array} \right\}$$

P R O P.

PROP. III.

229. The Expression for the Superficies of the second *Tetraedron* in the Terms of the first circumscribing *Sphere*, is $\frac{8 R^2}{9\sqrt{3}}$.

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} \frac{8 R^2}{3} = \\ \frac{8 R^2}{9 \times 3} = \\ \frac{8 R^2}{\sqrt{3}} = \end{array} \right\} \begin{array}{l} \text{Expression} \\ \text{for the} \\ \text{Sq. of} \\ \text{the} \\ \text{side} \\ \text{of} \end{array} \left\{ \begin{array}{l} 1 \text{ ft} \\ 2 \text{ d} \end{array} \right\} \left\{ \begin{array}{l} \text{Te-} \\ \text{trae-} \\ \text{dron,} \end{array} \right\} \begin{array}{l} \text{p. ar. 109} \\ \text{p. ar. 226.} \end{array}$

But $\left\{ \begin{array}{l} 4 \end{array} \right\} \left\{ \begin{array}{l} \frac{8 R^2}{3} : \frac{8 R^2}{9 \times 3} :: \frac{8 R^2}{\sqrt{3}} : \frac{8 R^2}{9\sqrt{3}} \end{array} \right\} \begin{array}{l} \text{Sup. of 1 ft Tetraed. p. 115} \\ \text{p. 16.6 \& Art. 106} \end{array}$

THEOREM.

Divide the Cube of the Diameter of the first circumscribing *Sphere*, by 9 times the Square Root of 3; and the Quotient is the Superficies of the second *Tetraedron*.

PROP.

PROP. VI.

230. The Expression for the Solidity of the second *Tetraedron* in the Terms of the first circumscribing Sphere, is $\frac{8 R^3}{243\sqrt{3}}$.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{16R^3\sqrt{2}}{3\sqrt{3}} = \right. \\ 2 \left| \frac{16R^3\sqrt{2}}{81\sqrt{3}} = \right. \\ 3 \left| \frac{8R^3}{9\sqrt{3}} = \right. \\ 4 \left| \frac{16R^3\sqrt{2}}{3\sqrt{3}} : \frac{16R^3\sqrt{2}}{81\sqrt{3}} :: \frac{8R^3}{9\sqrt{3}} : \end{array} \right. \left. \begin{array}{l} \text{Cube of} \\ \text{the side} \\ \text{of the} \end{array} \left\{ \begin{array}{l} 1\text{ft} \\ 2\text{d} \end{array} \right\} \text{Tetra} \left\{ \begin{array}{l} 109 \\ p. ar. \\ 223 \end{array} \right. \right. \\ \text{But} \left\{ \begin{array}{l} \frac{8R^3}{243\sqrt{3}} \left\{ \begin{array}{l} \text{Sol. of the 1st Tetraed.} \\ \text{Solid. of the 2d Te-} \end{array} \right. \\ \text{and Art. 105.} \end{array} \right. \end{array}$$

THEOREM.

Divide the *Cube* of the first circumscribing *Sphere's* Diameter by 243 times the Square Root of 3; and the Quotient will be the Solidity of the second *Tetraedron*.

Sect. II. Of the HEXAEDRON.

PROP. I.

231. The Diameter of a *Sphere* inscribed within a *Cube*, or *Hexaedron*, is expressed by $\frac{2R}{3}$, in the Terms of the circumscribing *Sphere*.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. & \left\{ \begin{array}{l} R^2 = \\ 2 R^2 = \\ \frac{R^2}{3} = \text{Sq. of the Rad. of} \end{array} \right. \left\{ \begin{array}{l} \text{Square} \\ \text{Radius} \\ \text{of the} \end{array} \right. \left\{ \begin{array}{l} \text{Sphere} \\ \text{Circle} \end{array} \right\} \left\{ \begin{array}{l} \text{per Art.} \\ \text{per Art.} \end{array} \right. \\
 & & \left\{ \begin{array}{l} 109 \\ 123 \end{array} \right. \\
 1-2 & & \left\{ \begin{array}{l} \frac{R^2}{3} = \text{Sq. of the Rad. of} \\ \text{inscribed Sphere, per} \end{array} \right\} 222. \\
 3 \text{ or } 2 \times 2 & 4 & \frac{2R}{\sqrt{3}} = \text{Diameter of the said Sphere.}
 \end{array}$$

THEOREM.

Divide the Diameter of the circumscribing *Sphere*, by the Square Root of 3, and the Quotient will be the Diameter of the inscribed *Sphere*.

COROLLARY.

232. If there be a *Tetraedron* and a *Cube*, both inscribed in the same *Sphere*, the Height of the *Tetraedron*, will be to the Diameter of a *Sphere* inscribed within the *Cube*, as the side of the *Cube*, is to the Radius of the circumscribing *Sphere*.

O

Height

$$\left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} \frac{2x}{\sqrt{3}} \\ \frac{2R}{\sqrt{3}} \\ \frac{2x}{\sqrt{3}} : \frac{2R}{\sqrt{3}} \end{array} \begin{array}{l} = \text{Height of the Tetraedron,} \\ = \text{Diam. of the inscr. Sphere,} \\ :: x : R, \text{ per 16.6.} \end{array} \begin{array}{l} 196. \\ 231. \end{array}$$

233. From this, and *Art.* 122. it is plain, that the Diameter of a *Sphere* inscribed within a *Cube*, is equal to the side of the *Cube*; which that it is so, will be evident from a bare Inspection of the *Cube*.

COROLLARY.

234. The Diameter of the circumscribing *Sphere*, is in power to the Diameter of the inscribed *Sphere*, as 3 to 1.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} 4R^2 \\ 4h^2 \\ 3 \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} \text{Sq Diam. of the } = \left\{ \begin{array}{l} \text{circumsf.} \\ \text{inscrib'd} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{Sphere,} \end{array} \right\} \text{ per Art. } \left\{ \begin{array}{l} 109 \\ 231 \end{array} \right\} \\ \text{But } \left| \begin{array}{l} 3 \end{array} \right| 4R^2 : \frac{4R^2}{3} :: 3 : 1, \text{ per 16.6.} \end{array}$$

PROP. II.

235. The Expression for the side of a *Cube* inscribed within this inscribed *Sphere*, in the Terms of the first circumscribing *Sphere*, is $\frac{2R}{3}$.

For

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{4R^2}{3} = \text{Square of the Diameter of the} \right. \\ \quad \quad \quad \text{inscribed Sphere} \quad \quad \quad 231 \\ 2 \left| 3 : 1 :: \frac{4h^2}{3} : \frac{4h^2}{9} = \text{Sq. of the side} \right. \\ \quad \quad \quad \text{of the Cube} \quad \quad \quad 122. \\ 2 \text{ or } 2 \left| 3 \left| \frac{2R}{3} = \text{side of the 2d Cube} \right. \right. \end{array} \right.$$

THEOREM.

Take one third part of the first circumscribing Sphere's Diameter, for the side of the second Cube.

COROLLARIES.

236. From this and Art. 223, it appears, that the side of the second Cube is equal to the Diameter of a Sphere inscribed in a Tetraedron, that is inscribed in the same Sphere, in which the first Cube was inscribed.

237. If a Cube and a Tetraedron be both inscribed in the same Sphere, the Diameter of a Sphere inscribed in the Cube, is in power to the Diameter of the Sphere inscribed in the Tetraedron, as 3 to 1.

$$\text{For } \left\{ \begin{array}{l} 1 \left| \frac{4R^2}{3} \right\} = \text{Sq. of the} \\ \quad \quad \quad \text{Diamet. of} \left\{ \begin{array}{l} \text{Cube.} \quad 231 \\ \text{the Sphere} \\ \text{insc. in the} \left\{ \begin{array}{l} \text{Tetraedron} \quad 223 \end{array} \right. \end{array} \right. \\ 2 \left| \frac{4R^2}{9} \right\} \end{array} \right.$$

But $3 \left| \frac{4R^2}{3} : \frac{4R^2}{9} :: 3 : 1, \text{ per 16.6.} \right.$

O 2

238. The

238. If a *Cube* and a *Tetraedron* be both inscribed in the same Sphere, the side of the Cube will be a mean Proportional between the Diameter of the circumscribing Sphere, and the Diameter of the Sphere inscribed within the *Tetraedron*.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{2R}{\sqrt{3}} = \text{side of the Cube} \right. \\ 2 \left| \frac{2R}{2} = \text{Diam. of the Sphere} \right. \\ 3 \left| \frac{2R}{3} = \text{Diam. of the inscrib. Sphere} \right. \end{array} \right\} \text{per Art.} \left\{ \begin{array}{l} 122 \\ 109 \\ 223 \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 4 \left| 2R : \frac{2R}{\sqrt{3}} :: \frac{2R}{\sqrt{3}} : \frac{2R}{3}, \text{ per 16.6.} \right. \end{array} \right. \end{array}$$

P R O P. III.

239. The Expression for the Superficies of the second *Cube* in the Terms of the first circumscribing Sphere, is $\frac{8R^2}{3}$.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{2R}{3} = \text{side of this Cube per Art. 235} \right. \\ 2 \left| \frac{4R^2}{9} = \text{Superficies of one Face.} \right. \\ 3 \left| \frac{24R^2}{9} = \frac{8R^2}{3} = \text{whole Superficies.} \right. \end{array} \right. \\ 1 \odot 2 \\ 2 \times 3 \end{array} \text{per Art. 37.}$$

T H E O R E M.

Divide the Square of the double Diameter of the first circumscribing Sphere by 3, the Quotient will be the Superficies of the second *Cube*, in the Terms of the first circumscribing Sphere.

C O R-

COROLLARIES.

240. From this and *Art.* 109, it appears, that the Superficies of this *Cube*, is equal to the Square of the side of a *Tetraedron* inscribed in the first circumscribing Sphere.

241. The Superficies of this *Cube*, is to the Square of the Radius of a Circle encompassing the Triangle of the first *Tetraedron*, as 3 to 1.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{8R^2}{3} = \text{Superficies of this Cube} \quad 139 \\ 2 \left| \frac{8R^2}{9} = \text{Square of Radius, \&c.} \quad 110 \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{8R^2}{3} : \frac{8R^2}{9} :: 3 : 1, \text{ per 16.6.} \end{array} \right. \end{array}$$

PROP. IV.

242. The Expression for the Solidity of the second *Cube* in the Terms of the first circumscribing Sphere, is $\frac{8R^3}{27}$.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{2R}{3} = \text{side of this Cube.} \quad 235 \\ 2 \left| \frac{8R^3}{27} = \text{Solidity.} \end{array} \right. \\ 1 \odot 3 \end{array}$$

THEOREM.

Divide the *Cube* of the Diameter of the first circumscribing Sphere by 27, and the Quotient will be the Solidity of the second *Cube*.

COROL-

COROLLARY.

243. The Solidity of the first circumscribing Sphere is to the Solidity of this *Cube*, as $9e$ to 2.

$$\begin{array}{lcl} \text{For} & \left| \begin{array}{l} 1 \\ 2 \end{array} \right| & \left\{ \begin{array}{l} \frac{4 R^3 e}{3} \\ \frac{8 R^3}{27} \end{array} \right\} = \left\{ \begin{array}{l} \text{Solidity of the} \\ \text{Sphere} \\ \text{Cube} \end{array} \right\} \quad \begin{array}{l} 101 \\ 242 \end{array} \\ \text{But} & \left| \begin{array}{l} 3 \end{array} \right| & \frac{4 R^3 e}{3} : \frac{8 R^3}{27} :: 9e : 2, \text{ per 16.6} \end{array}$$

Sect. III. Of the OCTAEDRON.

PROP. I.

244. The Expression for the Diameter of a *Sphere*, inscribed in an *Octaedron*, in the Terms of the circumscribing Sphere, is $\frac{2 R}{\sqrt{3}}$.

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} & \left\{ \begin{array}{l} R^2 \\ \frac{2 R^2}{3} \end{array} \right\} = \left\{ \begin{array}{l} \text{Sq. of the} \\ \text{Rad. of the} \\ \text{circumscr.} \end{array} \right\} \left\{ \begin{array}{l} \text{Sphere} \\ \text{Circle} \end{array} \right\} \quad \begin{array}{l} 109 \\ 137 \end{array} \\ \text{I—2} & \left\{ \begin{array}{l} 3 \end{array} \right\} & \left\{ \begin{array}{l} R^2 \\ \frac{R^2}{3} \end{array} \right\} = \left\{ \begin{array}{l} \text{Sq. of the Radius of} \\ \text{the inscribed Sphere} \end{array} \right\} \quad 222 \\ 3 \text{ w } 2 \& \times 2 & \left\{ \begin{array}{l} 4 \end{array} \right\} & \frac{2 R}{\sqrt{3}} = \text{Diameter.} \end{array}$$

THEO-

Divide the Diameter of the circumscribing *Sphere* by the Square Root of 3, the Quotient will be the Diameter of the inscribed *Sphere*.

245. From this and *Art.* 231 it is evident, that if there be a *Cube* and an *Octaedron* inscribed in the same Sphere, the Diameter of the Sphere inscribed in each of them will be equal.

246. The Diameter of a Sphere circumscribing an *Octaedron*, is in power to the Diameter of a Sphere inscribed within it, as 3 to 1.

For	$\left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\}$	$\left \begin{array}{c} 4 R^2 \\ \frac{4 R^2}{3} \end{array} \right\}$	$\left. \begin{array}{l} = \text{Sq. of} \\ \text{the Diam.} \\ \text{of the} \end{array} \right\}$	$\left. \begin{array}{l} \text{circum.} \\ \text{inscrib.} \end{array} \right\}$	$\left. \begin{array}{l} \text{Sphere} \\ \text{244} \end{array} \right\}$	109
But						3

247. Also, from hence and *Article 122*, it is plain, that if a *Cube* and an *Octaedron* be both inscribed in one Sphere, the side of the *Cube* will be equal to the Diameter of the Sphere inscribed in the *Octaedron*.

248. The

248. The side of an *Octaedron*, is in power to the Diameter of its inscribed Sphere, as 3 to 2.

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} 4R^2 : Z^2 :: 2 : 1. \quad 135 \\ 4R^2 : \frac{4R^2}{3} :: 3 : 1. \quad 246 \end{array} \right. \\ \text{But} & \left| \begin{array}{l} 3 \end{array} \right. & \left| \begin{array}{l} Z^2 : \frac{4R^2}{3} :: 3 : 2 \end{array} \right. \end{array}$$

P R O P. II.

249. The Expression for the side of a *Cube* inscribed in an *Octaedron*, in the Terms of the Sphere circumscribing the *Octaedron*, is $\frac{2R}{3}$.

N. B. The *Cube* inscribed in an *Octaedron*, is that which is inscribed in its inscribed Sphere. For the side of this *Cube* put v .

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} \frac{4R^2}{3} = \text{Sq. of the Diam. of} \\ \text{the inscr. Sphere} \end{array} \right. \} \quad 244 \\ \text{And} & \left| \begin{array}{l} 2 \end{array} \right. & \left| \begin{array}{l} \frac{4R^2}{3} : v^2 :: 3 : 2. \quad 122 \end{array} \right. \\ 2vv2, \text{ \&c.} & \left| \begin{array}{l} 3 \end{array} \right. & \left| \begin{array}{l} \frac{2R}{3} = v = \text{side of the Cube} \end{array} \right. \end{array}$$

THE O-

THEOREM.

Divide the Diameter of a Sphere circumscribing an *Octaedron* by 3, and the Quotient will be the side of a *Cube* inscribed within that *Octaedron*.

COROLLARIES.

250. From this and *Articles* 223, 235, it appears, that if a *Tetraedron*, a *Cube*, and an *Octaedron*, be all inscribed in the same Sphere, the Diameter of a Sphere inscribed in the *Tetraedron*, the side of a *Cube* inscribed in a Sphere that is inscribed in the first *Cube*, and the side of a *Cube* inscribed in the *Octaedron* are all equal to one another.

251. If a *Cube* and an *Octaedron* be both inscribed in the same Sphere, and if in the *Octaedron* there be inscribed another *Cube*, the side of the first *Cube* shall be in power to the side of the second *Cube*, as 3 to 1.

$$\begin{array}{l} \text{For} \\ \text{But} \end{array} \left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \left| \begin{array}{l} \frac{4 R^2}{3} = x^2 \\ \frac{4 R^2}{9} = v^2 \\ \frac{4 R^2}{3} : \frac{4 R^2}{9} :: 3 : 1, \text{ per 16.6.} \end{array} \right. \begin{array}{l} 122 \\ 249 \end{array}$$

252. The same things being still supposed, I say that the Square of the side of the first *Cube*, is a mean Proportional betwixt the Square of the Diameter

meter of the first Sphere, and the Square of the side of the second *Cube*.

For $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 4R^2 : \frac{4R^2}{3} :: 3 : 1, \quad 234. \\ \frac{4R^2}{3} : \frac{4R^2}{9} :: 3 : 1, \quad 251 \end{array} \right.$

But $3 \left| 4R^2 : \frac{4R^2}{3} : \frac{4R^2}{3} : \frac{4R^2}{9}, \quad 11.5 \text{ or } (16.6$

PROP. III.

253. The Expression for the side of an *Octaedron* inscribed within the inscribed Sphere, or second *Octaedron*, in the Terms of the first circumscribing Sphere, is $R \times \sqrt{\frac{2}{3}}$.

For $\left\{ \begin{array}{l} 1 \quad \frac{4 R^2}{3} = \text{Square of the Diameter of the inscr. Sphere} \quad 244 \\ 2 \quad 2 : 1 :: \frac{4 R^2}{3} : \frac{2 R^2}{3} = \text{Square of the side, \&c.} \quad 135. \\ 3 \quad R \times \sqrt{\frac{2}{3}} = \text{side of the second Octaedron.} \end{array} \right.$

T H E O R E M.

Multiply the Radius of the first circumscribing Sphere, into the Square Root of two thirds of an Unit, and the Product will be the side of the second *Octaedron*.

PROP.

P R O P. IV.

254. The Expression for the Superficies of the second *Octaedron*, in the Terms of the first circumscribing Sphere, is $\frac{4 R^2}{\sqrt{3}}$.

For	$\left\{ \begin{array}{c c} 1 & 2 R^2 \\ 2 & \frac{2 R^2}{3} \end{array} \right\}$	$= \text{Sq. of the side of the}$	$\left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\}$	Octaed.	135
	3	$4 R^2 \sqrt{3} = \text{Superficies of the first Octaedron,}$			144
But	4	$2 R^2 : \frac{2 R^2}{3} :: 4 R^2 \sqrt{3} : \frac{4 R^2}{\sqrt{3}} = \text{Superficies, \&c.}$			106

P R O P. V.

255. The Expression for the Solidity of the second *Octaedron*, in the Terms of the first circumscribing Sphere, is $\frac{4 R^3}{9\sqrt{3}}$.

P 2

For

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. & \left| \begin{array}{l} 2 R^3 \sqrt{2} \\ 2 R^3 \sqrt{2} \\ \frac{4 R^3}{3} \end{array} \right. \begin{array}{l} = \text{Cube of} \\ \text{the side of} \\ \text{the} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \text{Octaed.} \begin{array}{l} 135 \\ 253 \end{array} \\
 & & \frac{4 R^3}{3} = \text{Solidity of the first} \quad 149 \\
 \text{But} & 4 & \left| \begin{array}{l} 2 R^3 \sqrt{2} : \frac{2 R^3 \sqrt{2}}{3 \sqrt{3}} :: \frac{4 R^3}{3} : \frac{4 R^3}{9 \sqrt{3}} \\ = \text{Solid. of the 2d Octaed.} \end{array} \right. \quad 105
 \end{array}$$

T H E O R E M.

Divide four times the *Cube* of the Radius of the first Sphere, by 9 times the Square Root of 3 and the Quotient will be the Solidity of the second *Octaedron*.

Or thus ;

Multiply the Area of this *Octaedron*, by a ninth part of the Radius of the first circumscribing Sphere, and that Product is the Solidity of the second *Octaedron*.

C O R O L L A R I E S.

256. The Solidity of the first circumscribing Sphere, is to the Solidity of this *Octaedron*, as $3 e \sqrt{3} : 1$.

For

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{4 R^3 e}{3} \right. \\ 2 \left| \frac{4 R^3}{9\sqrt{3}} \right. \end{array} \right\} = \text{Solid. of the} \left\{ \begin{array}{l} 1^{\text{st}} \text{ Sphere} \quad 101. \\ 2^{\text{d}} \text{ Octaedron} \quad 255 \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{4 R^3 e}{3} : \frac{4 R^3}{9\sqrt{3}} \right. \end{array} \right\} :: 3 e\sqrt{3} : 1, \text{ per 16.6.} \end{array}$$

257. The Solidity of the first *Cube* is to the Solidity of this second *Octaedron*, as 6 to 1.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{8 R^3}{3\sqrt{3}} \right. \\ 2 \left| \frac{4 R^3}{9\sqrt{3}} \right. \end{array} \right\} = \text{Sol. of the} \left\{ \begin{array}{l} 1^{\text{st}} \text{ Cube} \quad 129 \\ 2^{\text{d}} \text{ Octaedron} \quad 255 \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{8 R^3}{3\sqrt{3}} : \frac{4 R^3}{9\sqrt{3}} \right. \end{array} \right\} :: 6 : 1, \text{ per 16.6.} \end{array}$$

258. The Solidity of the first *Tetraedron*, is to the second *Octaedron*, as 2 to 1.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{8 R^3}{9\sqrt{3}} \right. \\ 2 \left| \frac{4 R^3}{9\sqrt{3}} \right. \end{array} \right\} = \text{Sol. of the} \left\{ \begin{array}{l} 1^{\text{st}} \text{ Tetraedron} \quad 117 \\ 2^{\text{d}} \text{ Octaedron} \quad 255 \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{8 R^3}{9\sqrt{3}} : \frac{4 R^3}{9\sqrt{3}} \right. \end{array} \right\} :: 2 : 1, \text{ per 16.6.} \end{array}$$

Sect.

Sect. IV. *Of the DODECAEDRON.*

P R O P. I.

259. The Expression for the Diameter of a Sphere inscribed within a *Dodecaedron*, in the Terms of its circumscribing Sphere, is $\frac{R}{S} \times \sqrt{4S^2 - b}$

For the Diameter is nothing but the double Distance from the middle or Center of the Body, to the Center of one of the *Pentagonal* Faces, as is very evident from the Body it self, and is therefore the Expression above, *per Art.* 163.

T H E O R E M.

Multiply the Square Root of four times the Square of the Sine of 36 Degrees, made less by b , into the Radius of the circumscribing Sphere, and divide that Product by the Sine of 36 Degrees, the Quotient will be the Diameter of the inscribed Sphere.

C O R O L L A R I E S.

260. The Diameter of a Sphere circumscribing a *Dodecaedron*, is in power to the Diameter of a Sphere inscribed within it, as 4 to $4 - \frac{b}{S^2}$.

For

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 4 R^2 \\ \frac{R^2}{S^2} \times 4 S^2 - b \end{array} \right\} = \text{Sq. of the Dia. of} \\ \left\{ \begin{array}{l} \text{circumscr.} \\ \text{the inscribed} \end{array} \right\} \text{Sphere} \left\{ \begin{array}{l} 109 \\ 259 \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 3 \end{array} \right. \left| \begin{array}{l} 4 R^2 : \frac{R^2}{S^2} \times 4 S^2 - b :: 4 : 4 - \frac{b}{S^2} \\ \text{per 16.6.} \end{array} \right. \end{array}$$

261. The side of a *Dodecaedron*, is in power to the Diameter of the inscribed Sphere, as $\frac{b}{4S^2 - b}$ to $\frac{1}{S^2}$.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} R^2 b \\ \frac{R^2}{S^2} \times 4 S^2 - b \end{array} \right\} = \text{Sq. of} \left\{ \begin{array}{l} \text{side of Dode. 155} \\ \text{Dia. of inscr. 259} \\ \text{(Sphere)} \end{array} \right. \\ \text{But} \left\{ \begin{array}{l} 3 \end{array} \right. \left| \begin{array}{l} R^2 b : \frac{R^2}{S^2} \times 4 S^2 - b :: \frac{b}{4 S^2 - b} \\ : \frac{1}{S^2}, p. 16.6. \end{array} \right. \end{array}$$

P R O P. II.

262. The Expression for the side of a *Dodecaedron*, inscribed within the inscribed Sphere, or second *Dodecaedron*, in the Terms of the first circumscribing Sphere, is $\frac{R}{2S} \times \sqrt{4S^2 b - b^2}$.

For

$$\begin{array}{lcl}
 \text{For} & \left| \begin{array}{c} 1 \\ \hline \end{array} \right| & \frac{R}{S} \times \sqrt{4S^2 - b} = \text{Diam. \&c.} \quad 259 \\
 \text{But} & \left| \begin{array}{c} 2 \\ \hline \end{array} \right| & 2R : R\sqrt{b} :: \frac{R}{S} \times \sqrt{4S^2 - b} : \frac{R}{2S} \\
 & & \times \sqrt{4S^2b - b^2} = \text{fide per 4.6.}
 \end{array}$$

T H E O R E M.

Multiply four times the Square of the Sine of 36 Degrees into b , from that Product subtract the Square of b , then the Square Root of the Remainder multiplied into the Radius of the first circumscribing Sphere, and that Product divided by twice the Sine of 36 Degrees, will be the Side of the second *Dodecaedron*.

P R O P. III.

263. The Expression for the Height of a *Pentagonal Pyramid* of the second *Dodecaedron*, in the Terms of the first circumscribing Sphere, is $\frac{R}{4S^2} \times \sqrt{4S^2 - b}$.

For

259 R 2 S 5.	For	1	$\frac{R^2}{S^2} \times 4S^2 - b = \text{Sq. of the Diam.}$ of the second Sphere, 259
2 w 2	But	2	$4 : \frac{4S^2 - b}{4S^2} :: \frac{R^2}{S^2} \times 4S^2 - b :$ $\frac{R^2}{16S^2} \times 4S^2 - b^2, p. 16.6. 163$
		3	$\frac{R}{4S^2} \times 4S^2 - b = \text{Height of the Pen-}$ tagonal Pyramid.

T H E O R E M.

From 4 times the Square of the Sine of 36 Degrees, subtract b ; multiply the Remainder into the Square of the Radius of the first circumscribing Sphere; divide that Product by twice the Sine of 36 Degrees, and the Quotient will be the Height of a Pyramid of the second *Dodecaedron*.

C O R O L L A R Y.

264. The Radius of the Sphere inscribed in a *Dodecaedron*, is a Mean Proportional betwixt the Radius of the first circumscribing Sphere, and Height of the Pyramid of the second *Dodecaedron*.

Q

For

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left\{ \begin{array}{l} R \\ \frac{R}{2S} \times \sqrt{4S^2 - b} \end{array} \right\} = \text{Rad. of the } \left\{ \begin{array}{l} 1^{\text{st}} \\ 2^{\text{d}} \end{array} \right\} \text{ Sphere} \\
 & & \left\{ \begin{array}{l} 109 \\ 259 \end{array} \right\} \\
 & \left\{ \begin{array}{l} 3 \\ 4 \end{array} \right. & \left\{ \begin{array}{l} \frac{R}{4S^2} \times \sqrt{4S^2 - b} = \text{Height of the Pyra-} \\ \text{mid} \end{array} \right\} \\
 & & 263 \\
 \text{But} & \left\{ \begin{array}{l} 4 \end{array} \right. & \left\{ \begin{array}{l} R : \frac{R}{2S} \times \sqrt{4S^2 - b} :: \frac{R}{2S} \times \sqrt{4S^2 - b} \\ : \frac{R}{4S^2} \times \sqrt{4S^2 - b}, \text{ per 16.6.} \end{array} \right\}
 \end{array}$$

P R O P. IV.

265. The Expression for the Superficies of a *Dodecaedron* inscribed in the inscribed Sphere or second *Dodecaedron*, in the Terms of the first circumscribing Sphere, is $\frac{5R^2}{4S^3} \times \sqrt{4S^2 - b}$.

For

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left\{ \begin{array}{l} R^2 \\ \frac{R^2}{4S^2} \times \overline{4S^2 - b} \end{array} \right\} = \text{Sq. of Rad. of the} \\
 & & \left\{ \begin{array}{l} 1^{\text{st}} \\ 2^{\text{d}} \end{array} \right\} \text{ Sphere} \left\{ \begin{array}{l} 109 \\ 259 \end{array} \right. \\
 & 3 & \frac{5R^2}{sS} = \text{Superf. of the 1}^{\text{st}} \text{ Dodecaed. } 168 \\
 \text{But} & 4 & R^2 : \frac{R^2}{4S^2} \times \overline{4S^2 - b} :: \frac{5R^2}{sS} : \\
 & & \frac{5R^2}{4sS^3} \times \overline{4S^2 - b} = \text{Superf. } 107
 \end{array}$$

T H E O R E M.

Divide five times the Square of the Radius of the first Sphere, by four times the Sine of 54 Degrees, multiplied into the *Cube* of the Sine of 36 Degrees; then multiply that Quotient into four times the Square of the Sine of 35 Degrees, made less by *b*, and the Product will be the Superficies of the *Dodecaedron*.

P R O P. V.

266. The Expression for the Solidity of the second *Dodecaedron*, in the Terms of the first circumscribing Sphere, is $\frac{5R^3}{48sS^5} \times \overline{4S^2 - b}^2$

Q 2

For

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} \frac{5 R^2}{4 S S^3} \times \overline{4 S^2 - b} = \text{Superf.} \\ \frac{R}{12 S^2} \times \overline{4 S^2 - b} = \frac{n}{3} \end{array} \right. & \begin{array}{l} 265 \\ 263 \end{array} \\
 1 \times 2 & \left| \begin{array}{l} 3 \\ \frac{5 R^3}{48 S S^5} \times \overline{4 S^2 - b} = \text{Solidity} \end{array} \right. & 98
 \end{array}$$

The last another way.

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} 8 R^3 \\ \frac{R^3}{S^3} \times \overline{4 S^2 - b}^{\frac{1}{2}} \end{array} \right. & \left. \right\} = \text{Cube of Diam. of} \\
 & \left| \begin{array}{l} \text{the } \left\{ \begin{array}{l} 1 \text{ ft} \\ 2 \text{ d} \end{array} \right\} \text{ Sphere} \end{array} \right. & \left\{ \begin{array}{l} 109 \\ 259 \end{array} \right. \\
 3 & \left| \begin{array}{l} \frac{5 R^3}{6 S S^2} \times \sqrt{4 S^2 - b} = \text{Solidity of the} \\ \text{first Sphere.} \end{array} \right. & \\
 4 & \left| \begin{array}{l} 8 R^3 : \frac{R^3}{S^3} \times \overline{4 S^2 - b}^{\frac{1}{2}} :: \frac{5 R^3}{6 S S^2} \times \\ \sqrt{4 S^2 - b} : \frac{5 R^3}{48 S S^5} \times \overline{4 S^2 - b}^2 \\ = \text{Solidity.} \end{array} \right. &
 \end{array}$$

THEOREM.

From four times the Square of the Sine of 36 Degrees, subtract b ; and multiply the Square of the Remainder, into five times the *Cube* of the Radius of the first Sphere; divide that Product by 48 times the Sine of 54 Deg. multiplied into the fifth Power of the Sine of 36 Deg. and the Quotient will be the Solidity of the second *Dodecaedron*.

Sect.

Sect. V. Of the ICOSAEDRON.

PROP. I.

267. The Expression for the Diameter of a Sphere, inscribed in an *Icosaedron* in the Terms of the circumscribing Sphere, is $\frac{R}{S} \sqrt{4S^2 - b}$.

This Expression is the same with the like Expression for the *Dodecaedron*, as is evident from *Art.* 185.

COROLLARIES.

268. The Diameter of a Sphere circumscribing an *Icosaedron*, is in power to the Diameter of a Sphere inscribed in it, as 4 to $4 - \frac{b}{S^2}$.

This is the same with *Art.* 260, for the *Dodecaedron*.

PROP. II.

269. The Expression for the side of an *Icosaedron* inscribed in the inscribed Sphere, or second *Icosaedron*, in the Terms of the first circumscribing Sphere, is $\frac{R}{S} \times \sqrt{\frac{4S^2 - b}{5 \times ,5 + \sqrt{,05}}}$.

For

For	{	1	R	}	$=$	Radius of the	{	Sphere	{	109				
		2	$\frac{R}{2S} \times \sqrt{4S^2 - b}$								}	circumscr.	}	267
		3	$\frac{2R}{\sqrt{5 \times ,5 + \sqrt{,05}}}$											
But	{	4	$R : \frac{R}{2S} \times \sqrt{4S^2 - b} :: \frac{2R}{\sqrt{5 \times ,5 + \sqrt{,05}}}$	}	}	}	}	}	}					
		5	$\frac{R}{S} \times \sqrt{\frac{4S^2 - b}{5 \times ,5 + \sqrt{,05}}}$							}	}	}	}	}

T H E O R E M.

Multiply 5 into the Sum of ,5 more the Square Root of ,05; by that Product divide 4 times the Square of the Sine of 36 Degrees made less by b ; then multiply the Square Root of that Quotient into the Quotient of the Radius of the first Sphere, divided by the Sine of 36 Degrees, and that Product will be the side of the second *Icosaedron*.

The

270. The same another way.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} \frac{R^2}{S^2} \times \overline{4S^2 - b} = \text{Square of the Diam.} \\ \text{of the inscribed Sphere} \quad 267 \\ 16S^2 : 3b :: \frac{R^2}{S^2} \times \overline{4S^2 - b} : \frac{R^2}{16S^2} \\ \times \overline{12S^2b - 3b^2} \quad 180 \end{array} \right. \\
 \text{or} & 3 & \left| \begin{array}{l} \frac{R}{2S} \sqrt{12S^2b - 3b^2} = \frac{R}{2S} \sqrt{4S^2 - b} \\ \text{(per Art. 167.)} = \text{side, \&c.} \end{array} \right.
 \end{array}$$

THEOREM.

From four times the Square of the Sine of 36 Degrees, subtract b ; multiply the Square Root of the Remainder into the Radius of the circumscribing Sphere, and divide that Product by a double Rectangle under the Sine of 54 Degrees, and Sine of 36 Degrees, the Quotient will be the side of the second *Icosaedron*.

PROP. III.

271. The Expression for the Superficies of the second *Icosaedron*, in the Terms of the first circumscribing Sphere, is $\frac{15bR^2}{16S^2} \sqrt{3} \times \overline{4S^2 - b}$

For

	1	R^2	} = Sq. of the Diam. of	
	2	$\frac{R^2}{4S^2} \times \overline{4S^2 - b}$		
For			} circumscr.	109
				} inscribed
	3	$\frac{15bR^2}{4S^2} \sqrt{3}$	= Superficies of the Ico-	
			faedron	184
But	4	$R^2 : \frac{R^2}{4S^2} \times \overline{4S^2 - b} :: \frac{15bR^2}{4S^2} \sqrt{3} :$		
		$\frac{15bR^2 \sqrt{3}}{16S^4} \times \overline{4S^2 - b}$	= Super-	
		ficies, &c.		107.

T H E O R E M.

Multiply $15b$, the Square of the Radius of the first Sphere, the Square Root of 3, and four times the Sine of 36 Degrees less b , continually one into another; and divide the last Product by 16 times the Biquadrate of the Sine of 36 Degrees, and the Quotient will be the Superficies of the second *Icofaedron*.

P R O P. IV.

272. The Expression for the Solidity of the second *Icofaedron*, in the Terms of the first circumscribing Sphere, is $\frac{5bR^3 \sqrt{3}}{64S^3} \times \overline{4S^2 - b}^2$

For

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \left| \frac{15b R^2 \sqrt{3}}{16 S^4} \times 4S^2 - b = \text{Superf. } 271 \\ 2 \left| \frac{R}{12 S} \times 4S^2 - b = \frac{n}{3} \quad 263 \\ 3 \left| \frac{5b R^3 \sqrt{3}}{64 S^5} \times 4S^2 - b \right|^2 = \text{Solid. } 98 \end{array} \right. \\ 1 \times 2 \end{array}$$

T H E O R E M.

From four times the Square of the Sine of 36 Degrees subtract b , multiply the Square of the Remainder, $5b$, the Cube of the Radius of the first Sphere, and Square Root of 3, continually one into another; divide the last Product by 64 times the fifth Power of the Sine of 36 Degrees, and the Quotient will be the Solidity of the second *Icosædron*.



C H A P. IV.

Of the SPHERE, which passeth through the middle of each side of any of the Platonic Bodies.

273. **F**OR finding the Diameter of a Sphere which passeth through the middle of each side of these Bodies take this

G E N E R A L R U L E.

From the Square of the Radius of the circumscribing Sphere, subtract the Square of half the side of any of these Bodies, and the Square Root of the Remainder is the Radius of the Sphere, which passeth through the middle of each side of that Body, which doubled is the Diameter.

This is grounded upon the 47. 1. *Euclide*; for it is easy to conceive that the Radius of this Sphere is the Base of a Right-angled Triangle, whose Hypotenuse is the Radius of the circumscribing Sphere, and its Perpendicular is half the side of the Body, and therefore, &c.

Sect.

Sect. I. Of the TETRAEDRON.

PROP. I.

274. The Expression for the Diameter of a Sphere, passing through the middle of each side of a *Tetraedron*, in the Terms of the circumscribing Sphere, is $\frac{2R}{\sqrt{3}}$.

For	$\left\{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right\}$	$\left\{ \begin{array}{c} R^2 \\ \frac{2R^2}{3} \end{array} \right\}$	= Sq. of	$\left\{ \begin{array}{c} \text{Rad. of circum. sphere} \\ \text{of the} \end{array} \right\}$	109	
					$\left\{ \begin{array}{c} \frac{1}{2} \text{ side of the Tetraed.} \end{array} \right\}$	109
1—2			$\frac{R^2}{3}$	= Sq. of the Radius of the Sphere		
				passing, &c.		273
3—4 &c		$\frac{2R}{\sqrt{3}}$	= Diam. &c.			

THEOREM.

Divide the Diameter of the Sphere which circumscribeth a *Tetraedron*, by the Square Root of 3, and the Quotient will be the Diameter of the Sphere passing through the middle of each side.

COROLLARIES.

275. The Diameter of the circumscribing Sphere is in power to the Diameter of this Sphere, as 3 to 1.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left| \begin{array}{l} 4 R^2 \\ \frac{4 R^2}{3} \end{array} \right\} = \text{Square } \left\{ \begin{array}{l} \text{circumf.} \\ \text{of the} \\ \text{Diam. of} \end{array} \right\} \left\{ \begin{array}{l} \text{this} \\ \text{Sphere} \end{array} \right\} \left\{ \begin{array}{l} 109 \\ 274 \end{array} \right\}$$

$$\text{But } \left| \begin{array}{l} 3 \\ 4 R^2 : \frac{4 R^2}{3} :: 3 : 1, p. 16.6 \end{array} \right|$$

276. The Diameter of the Sphere inscribed in a Tetraedron, is in power to the Diameter of a Sphere, passing through the middle of each side thereof, as 1 to 3.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left| \begin{array}{l} \frac{4 R^2}{9} \\ \frac{4 R^2}{3} \end{array} \right\} = \text{Sq. } \left\{ \begin{array}{l} \text{the inscr.} \\ \text{of the} \\ \text{Diam.} \end{array} \right\} \left\{ \begin{array}{l} \text{this} \\ \text{Sphere} \end{array} \right\} \left\{ \begin{array}{l} 223 \\ 274 \end{array} \right\}$$

$$\text{But } \left| \begin{array}{l} 3 \\ \frac{4 R^2}{9} : \frac{4 R^2}{3} :: 1 : 3, \text{ per } 16.6 \end{array} \right|$$

277. The Diameter of a Sphere, passing through the middle of each side of a Tetraedron, is a mean Proportional betwixt the Diameters of the circumscribing and inscribed Spheres.

For

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \middle| \begin{array}{l} 2R \\ 2R \\ \frac{\sqrt{3}}{2}R \end{array} \right\} = \text{Diam. of } \left\{ \begin{array}{l} \text{circumf.} \\ \text{this} \\ \text{incrib'd} \end{array} \right\} \text{ Sphere } \left\{ \begin{array}{l} 109 \\ 274 \\ 223 \end{array} \right.$$

$$\text{But } \left\{ \begin{array}{l} 4 \\ 2R \end{array} \right\} : \frac{2R}{\sqrt{3}} :: \frac{2R}{\sqrt{3}} : \frac{2R}{3} \text{ per 16.6.}$$

278. The side of a *Tetraedron* is in power to the Diameter of a Sphere passing through the middle of each side thereof, as 2 to 1.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \middle| \begin{array}{l} \frac{8R^2}{3} \\ \frac{4R^2}{3} \end{array} \right\} = \text{Sq. of the } \left\{ \begin{array}{l} \text{side of the Tetraed. 109} \\ \text{Diam. of the Sphere, \&c. 274.} \end{array} \right.$$

$$\text{But } \left\{ \begin{array}{l} 3 \\ \frac{8R^2}{3} \end{array} \right\} : \frac{4R^2}{3} :: 2 : 1, \text{ per 16.6}$$

279. The Expression for the side of a *Tetraedron* inscribed within this *Sphere*, in the Terms of the first circumscribing *Sphere*, is $\frac{2R}{3}\sqrt{2}$.

For

$$\text{For } \left\{ \begin{array}{l|l} 1 & \frac{4R^2}{3} = \text{Sq. of Diam. of this Sphere } 274 \\ 2 & 3:2::\frac{4R^2}{3}:\frac{8R^2}{9} = \text{Sq. of side } 109 \\ 3 & \frac{2R}{3}\sqrt{2} = \text{side of the 2d Tetraedron} \end{array} \right.$$

P R O P. III.

280. The Expression for the Superficies of a *Sphere* passing through the middle of each side of a *Tetraedron*, in the Terms of the first circumscribing *Sphere*, is $\frac{4R^2e}{3}$.

$$\text{For } \left\{ \begin{array}{l|l} 1 & \frac{4R^2}{3} \\ 2 & \frac{4R^2}{3} \end{array} \right\} = \text{Square of the Diameter of } \left\{ \begin{array}{l} \text{first} \\ \text{this} \end{array} \right\} \text{ Sphere } \left\{ \begin{array}{l} 109 \\ 274 \end{array} \right.$$

$$\text{But } \left\{ \begin{array}{l|l} 3 & 4R^2e = \text{Superf. of the first Sphere } 103 \\ 4 & 4R^2:\frac{4R^2}{3}::4R^2e:\frac{4R^2e}{3} = \text{Superficies, \&c. } 107 \end{array} \right.$$

T H E O-

THEOREM.

Take one third part of the Superficies of the first *Sphere*, for the Superficies of this *Sphere*.

COROLLARIES.

281. The Superficies of a *Sphere* circumscribing a *Tetraedron*, is to the Superficies of a *Sphere* passing through the middle of each side thereof, as the Square of the Diameter of this *Sphere*, is to the Square of the Diameter of the inscribed *Sphere*.

$$\text{For } \left\{ \begin{array}{l} 1 \left| 4 R^2 e : \frac{4 R^2 e}{3} :: 3 : 1. \right. \\ 2 \left| \frac{4 R^2}{3} : \frac{4 R^2}{9} :: 3 : 1. \right. \end{array} \right. \quad \begin{array}{l} 275 \\ 276 \end{array}$$

$$\text{But } \left\{ \begin{array}{l} 3 \left| 4 R^2 e : \frac{4 R^2 e}{3} :: \frac{4 R^2}{3} : \frac{4 R^2}{9} \right. \end{array} \right. p. 16.6$$

282. The Solidity of a *Sphere* circumscribed about a *Tetraedron*, is to the Superficies of a *Sphere* passing through the middle of each side, as the Radius of the circumscribing *Sphere* is to 1.

For

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{4R^3e}{3} \right. \\ 2 \left| \frac{4R^2e}{3} \right. \end{array} \right\} = \begin{cases} \text{Solidity, \&c.} & 101 \\ \text{Superficies, \&c.} & 280 \end{cases} \\ \text{But} \left\{ \begin{array}{l} 3 \left| \frac{4R^3e}{3} : \frac{4R^2e}{3} :: R : 1, \text{ per 16.6} \end{array} \right. \end{array}$$

P R O P. IV.

283. The Expression for the Solidity of the *Sphere*, passing through the middle of each side of a *Tetraedron*, in the Terms of the circumscribing *Sphere*, is $\frac{4R^3e}{9\sqrt{3}}$:

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \left| \frac{8R^3}{3\sqrt{3}} \right. \\ 2 \left| \frac{8R^3}{3\sqrt{3}} \right. \end{array} \right\} = \begin{cases} \text{Cube of the Diameter of} \\ \text{circumscr.} \\ \text{this} \end{cases} \left\{ \begin{array}{l} \text{Sphere} \end{array} \right\} \begin{cases} 109 \\ 274 \end{cases} \\ 3 \left| \frac{4R^3e}{3} \right. = \text{Solidity of the circumscri-} \\ \text{bing Sphere} \quad 101 \\ \text{But} \left\{ \begin{array}{l} 4 \left| \frac{8R^3}{3\sqrt{3}} : \frac{4R^3e}{3} :: \frac{4R^3e}{9\sqrt{3}}, \text{ (per} \right. \\ \left. 16.6) = \text{Solidity} \right. \end{array} \right\} \quad 104 \end{array}$$

The

284. The last thus ;

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \left| \frac{4 R^2 e}{3} = \text{Super.} \right. \\ 2 \left| \frac{R}{3\sqrt{3}} = \frac{1}{3} \text{Rad.} \right. \\ 3 \left| \frac{4 R^3 e}{9\sqrt{3}} = \text{Solid.} \right. \end{array} \right. & \left. \begin{array}{l} \text{of this} \\ \text{Sphere} \end{array} \right\} \\ 1 \times 2 & & \end{array} \quad \begin{array}{l} 280 \\ 274 \\ 98, \&c \end{array}$$

THEOREM.

Divide the Solidity of the circumscribing Sphere by 3 times the Square Root of 3, and the Quotient will be the Solidity of this Sphere.

PROP. V.

285. The Expression for the Superficies of a *Tetraedron* inscribed, within this Sphere, in the Terms of the first circumscribing Sphere, is $\frac{8 R^2}{3\sqrt{3}}$.

$$\text{For} \left\{ \begin{array}{l} 1 \left| \frac{8 R^2}{3} \right. \\ 2 \left| \frac{8 R^2}{9} \right. \\ 3 \left| \frac{8 R^2}{\sqrt{3}} \right. \end{array} \right\} = \begin{array}{l} \text{Sq. of} \\ \text{the side} \\ \text{of} \end{array} \left\{ \begin{array}{l} \text{the 1st} \\ \text{this} \end{array} \right\} \text{Tetraed} \left\{ \begin{array}{l} 109 \\ 179 \end{array} \right.$$

$$\text{But} \quad 4 \left| \frac{8 R^2}{3} : \frac{8 R^2}{9} :: \frac{8 R^2}{\sqrt{3}} : \frac{8 R^2}{3\sqrt{3}} = \text{Superf.} \&c \quad 106$$

THEO-

T H E O R E M.

Divide twice the Square of the Diameter of the first circumscribing Sphere by 3 times the Square Root of 3, and the Quotient will be the Superficies of this *Tetraedron*.

P R O P. VI.

286. The Expression for the Solidity of this *Tetraedron*, in the Terms of the first circumscribing Sphere, is $\frac{8 R^3}{81}$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \left| \frac{16 R^3 \sqrt{2}}{3\sqrt{3}} \right. \\ 2 \left| \frac{16 R^3 \sqrt{2}}{27} \right. \end{array} \right\} & = \text{Cube of the side of} \\
 & \left\{ \begin{array}{l} \text{the 1.} \\ \text{this} \end{array} \right\} \left\{ \begin{array}{l} \\ \text{Tetraed.} \end{array} \right\} & \begin{array}{l} 109 \\ 279 \end{array} \\
 & 3 \left| \frac{8 R^3}{9\sqrt{3}} \right. & = \text{Solid. of the 1st Tetraed. } 117 \\
 \text{But} & 4 \left| \frac{16 R^3 \sqrt{2}}{3\sqrt{3}} : \frac{16 R^3 \sqrt{2}}{27} :: \frac{8 R^3}{9\sqrt{3}} : \frac{8 R^3}{81} \right. & \\
 & & = \text{Solidity } 105
 \end{array}$$

T H E O-

THEOREM.

Divide the *Cube* of the Diameter of the first *Sphere* by 81, and the Quotient will be the Solidity of this *Tetraedron*.

COROLLARY.

287. From this Proposition it is evident, that as the Solidity of this *Tetraedron*, is to the Square of the first *Sphere's* Diameter, so is the said Diameter to 81.

Sect. II. Of the HEXAEDRON.

PROP. I.

288. The Expression for the Diameter of a *Sphere*, passing through the middle of each side of a *Hexaedron*, in the Terms of the circumscribing

Sphere, is $\frac{2R}{\sqrt{3}} \sqrt{2}$.

S 2

For

For	{	1	R^2	} = Square of the	{	Radius of	}	109
		2	$\frac{R^2}{3}$			half side		
				the circ. <i>Sphere</i>				
				of the <i>Cube</i>				122
1—2		3	$\frac{2 R^2}{3}$	= Sq. of Radius				273
3 <i>u</i> & x 2		4	$\frac{2 R}{\sqrt{3}}$	$\sqrt{2} = 2 R \sqrt{\frac{2}{3}} =$ Diameter.				

T H E O R E M.

Multiply the Diameter of the circumscribing *Sphere*, into the Square Root of $\frac{2}{3}$, and the Product will be the Diameter of this *Sphere*.

C O R O L L A R I I S.

289. The Diameter of a *Sphere* circumscribing a *Cube*, is in power to the Diameter of a *Sphere* passing through the middle of each side of the *Cube*, as 3 to 2.

For	{	1	$4 R^2$	} = Sq.	{	circumf.	}	109
		2	$\frac{8 R^2}{3}$			of the		
				Dia. of		this		288
But		3	$4 R^2$	$8 R^2$				
			3	$3 : 2$, per 16.6				

290. The

290. The side of a *Cube* is in power to the Diameter of a *Sphere* passing through the middle of each side thereof, as 1 to 2.

For $\left\{ \begin{array}{l} 1 \left| \frac{4 R^2}{3} \right. \\ 2 \left| \frac{8 R^2}{3} \right. \end{array} \right\} = \text{Sq. of the } \left\{ \begin{array}{l} \text{side of the } \textit{Cube} \text{ 122} \\ \text{Dia. of this } \textit{Sphere} \text{ 289} \end{array} \right.$

But $\left| \begin{array}{l} 3 \left| \frac{4 R^2}{3} \right. \\ 3 \left| \frac{8 R^2}{3} \right. \end{array} \right. : \frac{8 R^2}{3} :: 1 : 2, \text{ per 16.6}$

291. If there be a *Tetraedron* and a *Cube* inscribed in the same *Sphere*, the side of the *Tetraedron* will be equal to the Diameter of the *Sphere* which passeth through the middle of each side of the *Cube*.

This is evident from *Art.* 109, and *Art.* 288.

P R O P. II.

292. If there be a *Cube* and an *Octaedron* inscribed in the same *Sphere*, the Diameter of the circumscribing *Sphere* will have the same Proportion to the side of the *Octaedron*, which the Diameter of the *Sphere* passing through the middle of each side of the *Hexaedron* or *Cube* hath to the side of the *Cube*.

For

For	{	1	2 R =	{	Diameter of the		
	{	2	$\frac{2 R}{\sqrt{3}} \sqrt{2} =$	{	circumf.	{	109
But	{	3	$R \sqrt{2} =$	{	side of the	{	Octaedron 135
		5	$2 R : \frac{2 R}{\sqrt{3}} \sqrt{2} :: R \sqrt{2} : \frac{2 R}{\sqrt{3}}, \text{ per 16.6}$				

P R O P. III.

293. The Expression for the side of a *Cube* inscribed in the *Sphere*, which passeth through the middle of each side of the first *Cube*, in the Terms of the first circumscribing *Sphere*, is $\frac{2 R}{3} \sqrt{2}$.

For	{	1	$\frac{8 R^2}{3} = \text{Sq. of Dia. of this Sphere}$	288
	{	2	$3 : 1 :: \frac{8 R^2}{3} : \frac{8 R^2}{9} \text{ (per 16.6) =}$	122
2 nd	{	3	$\frac{2 R}{3} \sqrt{2} = \text{side of the 2d Cube}$	

T H E O-

T H E O R E M.

Multiply one third Part of the Diameter of the first circumscribing *Sphere*, into the Square Root of 2, and the Product will be the side of this *Cube*.

P R O P. IV.

294. The Expression for the Superficies of a *Sphere* passing through the middle of each side of a *Cube*, in the Terms of the first circumscribing

Sphere, is $\frac{8 R^2 e}{3}$.

For	{	1	$\frac{4 R^2}{3}$	} = Square of the Diameter of	
		2			
				{ circumf. }	} Sphere
		{ this — }		288	
		3	$4 R^2 e$	= Superficies of the first	
				<i>Sphere</i>	103
But		4	$4 R^2 : \frac{8 R^2}{3} :: 4 R^2 e : \frac{8 R^2 e}{3}$	= Su-	
			perficies, &c.		107

T H E O-

T H E O R E M.

Multiply the Superficies of the first Sphere by 2, and divide that Product by 3, and the Quotient will be the Superficies of this Sphere.

C O R O L L A R Y.

295. If the *Tetraedron* and *Cube* be both inscribed in the same Sphere, the Superficies of the Sphere passing through the middle of each side of the *Cube*, is to the Superficies of a *Sphere* passing through the middle of each side of the *Tetraedron*, as 2 to 1.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \left| \frac{8 R^2 e}{3} \right. \\ 2 \left| \frac{4 R^2 e}{3} \right. \end{array} \right\} & = \text{Superficies of the Sphere} \\
 & & \text{passing through the} \\
 & & \text{middle of the} \\
 & \left\{ \begin{array}{l} \text{Cube} \\ \text{Tetraedron} \end{array} \right\} & \\
 \text{But} & 3 \left| \frac{8 R^2 e}{3} : \frac{4 R^2 e}{3} :: 2 : 1, \text{ per 16.6.} \right. &
 \end{array}$$

P R O P. V.

296. The Expression for the Solidity of a *Sphere* which passeth through the middle of each side of the *Cube*, in the Terms of the first circumscribing

Sphere, is $\frac{8 R^3 e \sqrt{2}}{9 \sqrt{3}}$.

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \quad 8 R^3 \\ 2 \quad \frac{16 R^3 \sqrt{2}}{3 \sqrt{3}} \end{array} \right\} & = \text{Cube of the Diamet. of} \\
 & \left\{ \begin{array}{l} \text{the} \\ \text{other} \end{array} \right\} & \text{circumscribing} \\
 & & \text{Sphere} \left\{ \begin{array}{l} 109 \\ 288 \end{array} \right. \\
 \text{But} & \left\{ \begin{array}{l} 3 \quad \frac{4 R^3 e}{3} \\ 4 \quad 8 R^3 : \frac{16 R^3 \sqrt{2}}{3 \sqrt{3}} :: \frac{4 R^3 e}{3} : \frac{8 R^3 e \sqrt{2}}{9 \sqrt{3}} \end{array} \right\} & = \text{Solidity of the 1st Sphere 101} \\
 & & = \text{Solidity of the other Sphere, 105}
 \end{array}$$

T H E O R E M.

Multiply the Solidity of the first circumscribing *Sphere*, by twice the Square Root of 2, and divide that Product by three times the Square Root of 3, and the Quotient will be the Solidity of this *Sphere*.

T

Sect.

Sect. III. Of the OCTAEDRON.

P R O P. I.

297. The Expression for the Diameter of a Sphere, passing through the middle of each side of an *Octaedron*, in the Terms of the circumscribing Sphere, is $\frac{2R}{\sqrt{2}}$.

For	{	1	R^2	} = Sq. of {	Radius of the circum-	
		2	$\frac{R^2}{2}$			
					} scribing Sphere	109
						} <i>Octaedron</i>
1—2		3	$\frac{R^2}{2}$	= Sq. of the Radius		
3 <i>or</i> $\times 2$		4	$\frac{2R}{\sqrt{2}}$	= Diameter, &c.		

T H E O R E M.

Divide the Diameter of the Sphere circumscribing an *Octaedron* by the Square Root of 2, the Quotient will be the Diameter of a Sphere passing through the middle of each side thereof.

C O R O L.

COROLLARIES.

298. The Diameter of a *Sphere* passing through the middle of each side of an *Octaedron*, is in power to the Diameter of a *Sphere* circumscribing it, as 1 to 2.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left\{ \begin{array}{l} 4 R^2 \\ 2 R^2 \end{array} \right\} = \text{Sq. of Diam. } \left\{ \begin{array}{l} \text{circumscr. } 109 \\ \text{of Sphere. } \end{array} \right\} \text{passing, \&c. } 297 \\ \text{But } \left\{ \begin{array}{l} 1 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} 4 R^2 \\ 2 R^2 \end{array} \right\} : 4 R^2 :: 1 : 2, \text{ per 16.6.} \end{array}$$

299. If an *Octaedron* and a *Cube* be both inscribed in the same *Sphere*, the Diameter of the circumscribing *Sphere*, will be in power to the Diameter of a *Sphere* passing through the middle of each side of the *Octaedron*, as the Diameter of a *Sphere* passing through the middle of each side of the *Cube* is in power to the side of the *Cube*.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \left\{ \begin{array}{l} 4 R^2 \\ 2 R^2 \\ \frac{8 R^2}{3} \end{array} \right\} = \text{Sq. Diam. of } \left\{ \begin{array}{l} \text{circumscr. } 109 \\ \text{the Sphere } \left\{ \begin{array}{l} \text{Octaedron } 297 \\ \text{passing, \&c. } \end{array} \right\} \text{Cube } 288 \end{array} \right. \\ \left\{ \begin{array}{l} 4 \\ 5 \end{array} \right\} \left\{ \begin{array}{l} \frac{4 R^2}{3} \\ 4 R^2 \end{array} \right\} = \text{Sq. of the side of the Cube } 122 \\ \text{But } \left\{ \begin{array}{l} 1 \\ 5 \end{array} \right\} \left\{ \begin{array}{l} 8 R^2 \\ 4 R^2 \end{array} \right\} : \frac{8 R^2}{3} : \frac{4 R^2}{3}, \text{ per 16.6.} \end{array}$$

300. The side of an *Octaedron* is equal to the Diameter of a Sphere, which passeth through the middle of each side thereof.

This is evident from *Art.* 135, 297, and also from the Figure it self.

P R O P. II.

301. The Expression for the Superficies of the Sphere passing through the middle of each side of an *Octaedron*, in the Terms of the circumscribing Sphere, is $2 R^2 e$

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} & \left\{ \begin{array}{l} 4 R^2 \\ 2 R^2 \end{array} \right\} = \begin{array}{l} \text{Square of the Diameter of} \\ \text{the Sphere} \end{array} \\
 & & \left\{ \begin{array}{l} \text{circumscribing,} \\ \text{passing, \&c.} \end{array} \right. \begin{array}{l} 109 \\ 297 \end{array} \\
 \text{But} & \left\{ \begin{array}{l} 3 \\ 4 \end{array} \right\} & \left\{ \begin{array}{l} 4 R^2 e = \text{Superficies of the circum-} \\ \text{scribing Sphere} \end{array} \right. \begin{array}{l} 103 \\ 106 \end{array} \\
 & & 4 R^2 : 2 R^2 :: 4 R^2 e : 2 R^2 e = \text{Su-} \\
 & & \text{perficies, \&c.}
 \end{array}$$

The last otherways, thus ;

$$\begin{array}{lcl}
 \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} & \left\{ \begin{array}{l} 2 R^2 = \text{Square of the Diameter of} \\ \text{the Sphere, \&c.} \end{array} \right. \begin{array}{l} 297 \\ 103 \end{array} \\
 1 \times e & \left\{ \begin{array}{l} 2 \end{array} \right\} & \left\{ \begin{array}{l} 2 R^2 e = \text{Superficies, per Art.} \end{array} \right.
 \end{array}$$

T H E O-

THEOREM.

Multiply the Square of the Radius of the circumscribing *Sphere*, into twice the Ratio of the Diameter of a Circle to its Circumference, and the Product will be the Superficies of this *Sphere*.

COROLLARIES.

302. The Superficies of a *Sphere* circumscribing an *Octaedron*, is double to the Superficies of a *Sphere* passing through the middle of each side of it. This is plain from the last, for $4R^2e : 2R^2e :: 2 : 1$.

303. If a *Tetraedron*, a *Cube*, and an *Octaedron*, be all inscribed in the same *Sphere*, the Superficies of the circumscribing *Sphere*, is to the Superficies of the *Sphere* passing through the middle of each side of the *Octaedron*, as the Superficies of the *Sphere* passing through the middle of each side of the *Cube*, is to the Superficies of the *Sphere* passing through the middle of each side of the *Tetraedron*.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right\} \left\{ \begin{array}{l} 4 R^2 e \\ 2 R^2 e \\ 8 R^2 e \\ 3 \\ 4 R^2 e \\ 3 \end{array} \right\} = \text{Superficies of the Sphere} \left\{ \begin{array}{l} \text{circumscribing} \\ \text{passing, \&c.} \end{array} \right. \left\{ \begin{array}{l} \text{Octaed.} \\ \text{Cube} \\ \text{Tetraed.} \end{array} \right. \begin{array}{l} 103 \\ 301 \\ 294 \\ 280 \end{array} \\ \text{But} \left\{ \begin{array}{l} 5 \end{array} \right\} 4 R^2 e : 2 R^2 e :: \frac{8 R^2 e}{3} : \frac{2 R^2 e}{3\sqrt{2}} = \\ \text{per 16.6} \end{array}$$

PROP.

PROP. III.

304. The Expression for the Solidity of a Sphere passing thro' the middle of each side of an Octaedron, in the Terms of the circumscribing Sphere, is $\frac{2 R^3 e}{3\sqrt{2}}$.

For	{	1	$8 R^3$	}	=	{	Cube of the Diameter		
		2	$\frac{4 R^3 e}{3}$				Solidity		
		of the { circumscr. Sphere }					109 101		
		3	$\frac{4 R^3}{\sqrt{2}}$	=		Cube of the Diameter of the			
			<i>Sphere passing, &c.</i>				297		
But	{	4	$8 R^3 : \frac{4 R^3 e}{3} :: \frac{4 R^3}{\sqrt{2}} : \frac{2 R^3 e}{3\sqrt{2}}$		= Solidity, &c.		{	Solid.	
									105

The last otherways thus ;

For $\frac{R}{1 \times \frac{3\sqrt{2}}{1}}$	{	1	$2 R^2 e$	= Superf. of this Sphere	301
		2	$\frac{2 R^3 e}{3\sqrt{2}}$	= Solid. of this Sphere	297

THE O-

T H E O R E M.

Multiply the *Cube* of the Radius of the circumscribing Sphere, by twice the Ratio of the Diameter of a Circle to its Circumference ; and divide the Product by three times the Square Root of 2, and the Quotient will be the Solidity of this *Sphere*.

Sect. IV. *Of the DODECAEDRON.*

P R O P. I.

305. The Expression for the Diameter of a *Sphere*, passing through the middle of each side of a *Dodecaedron*, in the Terms of the circumscribing *Sphere*, is $2 R \times \sqrt{\frac{1}{2} + \frac{1}{6}\sqrt{5}}$.

For	{	1		R^2	} Square of the	} Radius of the			
		2		$\frac{R^2}{2} - \frac{R^2}{6} \sqrt{5}$			} of the	} half side of	
		circumscribing <i>Sphere</i>							109
		the <i>Dodecaedron</i>							155
1 — 2	3		$\frac{R^2}{2} + \frac{R^2}{6} \sqrt{5} = \text{Sq. of the Rad.}$	273					
3 — 2	4		$2 R \times \sqrt{\frac{1}{2} + \frac{1}{6}\sqrt{5}} = \text{Diameter of the}$	Sphere, &c.					

T H E O-

T H E O R E M.

To one sixth part of the Square Root of 5, add half an Unit; multiply the Square Root of that Sum into the Diameter of the circumscribing *Sphere*, and that Product will be the Diameter of this *Sphere*.

306. The last another way.

1	R^2	$\left. \begin{array}{l} \\ \\ \end{array} \right\} = \text{Sq. of the}$	$\left. \begin{array}{l} \text{Radius of the cir-} \\ \text{Half side of the } D\text{o-} \end{array} \right\}$	$\left. \begin{array}{l} \text{circumscribing Sphere} \\ \text{decaedron} \end{array} \right\}$	$\left. \begin{array}{l} 109 \\ 155 \end{array} \right\}$
2	$\frac{b R^2}{4}$				
1 — 2	3	$\frac{R^2}{4} \times 4 - b = \text{Sq. of the Diam.}$	273		
3 ω & $\times \frac{1}{2}$	4	$R \times \sqrt{4 - b} = \text{Sq. of the Dia. Sphere, \&c}$			

T H E O R E M.

From 4 subtract b , multiply the Square Root of the Remainder into the Radius of the circumscribing *Sphere*, and the Product will be the Diameter of the *Sphere* passing, &c.

C O R O L L A R I E S.

307. The Diameter of a *Sphere* circumscribing a *Dodecaedron*, is in power to the Diameter of a *Sphere* passing through the middle of each side thereof, as 1 to $\frac{1}{2} + \frac{1}{6} \sqrt{5}$. For

For	{	1		$4 R^2 = \text{Sq. of the Diam.}$	} of the cir-
		2		$4 R^2 e = \text{Superficies}$	
				bing <i>Sphere</i>	{ 109
					{ 103
But	{	3		$4 R^2 \times \frac{1}{2} + \frac{1}{6} \sqrt{5} = \text{Sq. of the Diam.}$	} of the <i>Sphere</i> passing, &c. 305
		4		$4 R^2 : 4 R^2 e :: 4 R^2 \times \frac{1}{2} + \frac{1}{6} \sqrt{5} :$	
				$4 R^2 e \times \frac{1}{2} + \frac{1}{6} \sqrt{5} = \text{Superfi-}$	105
				cies, &c.	

The last otherways.

1 x e	{	1		$4 R^2 \times \frac{1}{2} + \frac{1}{6} \sqrt{5} = \text{Square of the Dia-}$	} meter of the <i>Sphere</i> , &c. 305
		2		$4 R^2 e \times \frac{1}{2} + \frac{1}{6} \sqrt{5} = \text{Superficies, \&c.}$	
					(103

THEOREM.

To one sixth part of the Square Root of 5, add half an Unit ; multiply that Sum into the Superficies of the circumscribing *Sphere*, and the Product will be the Superficies of this *Sphere*.

310. The same another way.

1 x e	{	1		$R^2 \times \frac{1}{4} \sqrt{5} = \text{Square of the Diam.}$	} 306
		2		$R^2 e \times \frac{1}{4} \sqrt{5} = \text{Superficies}$	
					103

From

From 4 subtract b , multiply the Remainder into a fourth part of the Superficies of the circumscribing Sphere, and the Product will be the Superficies of this Sphere.

P R O P. III.

311. The Expression for the Solidity of a Sphere passing through the middle of each side of a Dodecaedron, in the Terms of the circumscribing Sphere, is

$$\frac{R^3}{6} e \times \overline{4-b}^{\frac{3}{2}}.$$

For	{	1	$8 R^3 = \text{Cube of the Diameter}$	}	{	109
		2	$\frac{4 R^3}{3} e = \text{Solidity.}$			101
		} of the circumf. Sphere				
But	{	3	$R^3 \times \overline{4-b}^{\frac{3}{2}} = \text{Cube of the Diameter of this Sphere}$	}	306	
		4	$8 R^3 : \frac{4 R^3}{3} e :: R^3 \times \overline{4-b}^{\frac{3}{2}} : \frac{R^3 e}{3} \times \overline{4-b}^{\frac{3}{2}} = \text{Solidity}$		105	

The last thus ;

For	{	1	$R^2 e \times \overline{4-b} = \text{Superf.}$	}	of this	310
		2	$\frac{R}{6} \times \sqrt{4-b} = \frac{1}{3} \text{ of Rad.}$			306
		} of this Sphere				
1 x 2	{	3	$\frac{R^3}{6} e \times \overline{4-b}^{\frac{3}{2}} = \text{Solidity}$		98	

THEOREM.

From 4 subtract b , multiply the *Cube* of the Square Root of that Remainder into one sixth part of the Product of the *Cube* of the Radius of the circumscribing *Sphere*, into the Ratio of the Diameter of a *Circle* to its Circumference, and that Product will be the Solidity of this *Sphere*.

Sect. V. Of the ICOSAEDRON.

PROP. I.

312. The Expression for the Diameter of a Sphere, passing through the middle of each side of an *Icosaedron*, is $\frac{R}{2S} \times \sqrt{16S^2 - 3b}$.

For	{	1		R^2	}	= Sq. of the	{	Radius of the					
1—2								2		$\frac{3bR^2}{16S^2}$	}	circumsf. <i>Sphere</i>	half side of
													3
3	1—2	2	1	$R^2 \times \frac{16S^2 - 3b}{16S^2}$		= Square of the		109					
						Radius, &c.		173					
3	$\frac{R}{2S} \times \sqrt{16S^2 - 3b}$	4		$\frac{R}{2S} \times \sqrt{16S^2 - 3b}$		= Diameter of		this <i>Sphere</i> .					

THE O-

THEOREM.

From 16 times the Square of the Sine of 36 Degrees, subtract $3b$; multiply the Remainder into the Radius of the circumscribing *Sphere*, and divide that Product by twice the Sine of 36 Degrees, the Quotient will be the Diameter, &c.

COROLLARIES.

313. The Diameter of a *Sphere* passing through the middle of each side of an *Icosaedron*, is in power to the Diameter of a *Sphere* circumscribing it, as

$$1 - \frac{3b}{16S^2} \text{ to } 1.$$

For	$\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right \begin{array}{l} R^2 \times \frac{16S^2 - 3b}{4S^2} \\ 4R^2 \end{array} \right\} = \text{Sq. of the Diam. of the Sphere}$		
		$\left\{ \begin{array}{l} \text{passing, \&c.} \\ \text{circumscr.} \end{array} \right.$	312 109
But		$3 \left R^2 \times \frac{16S^2 - 3b}{4S^2} : 4R^2 :: 1 - \frac{3b}{16S^2} : 1, \text{ per 16.6.} \right.$	

314. The side of an *Icosaedron* is in power to the Diameter of a *Sphere* passing through the middle of each side, as 1 to $\frac{16S^2}{3b} - 1$.

For

$$\begin{array}{l}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} \frac{3bR^2}{4S^2} = R^2 \times \frac{3b}{4S^2} \\ R^2 \times \frac{16S^2 - 3b}{4S^2} \end{array} \right\} = \text{Square of} \\
 \left. \begin{array}{l} \text{the} \\ \text{Diam. of the Sphere, \&c} \end{array} \right\} \begin{array}{l} \text{side} \\ \text{312} \end{array} \quad 180
 \end{array}$$

$$\begin{array}{l}
 \text{But } \left\{ \begin{array}{l} 3 \end{array} \right. \left| \begin{array}{l} R^2 \times \frac{3b}{4S^2} : R^2 \times \frac{16S^2 - 3b}{4S^2} :: 1 : \frac{16S^2 - 3b}{4S^2} \\ -1, \text{ (per 16.6)} \end{array} \right.
 \end{array}$$

315. The Diameter of a Sphere passing through the middle of each side of an *Icosaedron*, is in power to the Diameter of a Sphere inscribed within it, as $16S^2 - 3b : 16S^2 - 4b$.

$$\begin{array}{l}
 \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} R^2 \times \frac{16S^2 - 3b}{4S^2} \\ R^2 \times \frac{4S^2 - b}{S^2} \end{array} \right\} = \text{Square of the} \\
 \left. \begin{array}{l} \text{Dia. of Sphere} \\ \text{passing, \&c.} \\ \text{inscrib'd, \&c.} \end{array} \right\} \begin{array}{l} 312 \\ 267 \end{array}
 \end{array}$$

$$\begin{array}{l}
 \text{But } \left\{ \begin{array}{l} 3 \end{array} \right. \left| \begin{array}{l} R^2 \times \frac{16S^2 - 3b}{4S^2} : R^2 \times \frac{4S^2 - b}{S^2} \\ :: 16S^2 - 3b : 16S^2 - 4b, \text{ p. 16.6} \end{array} \right.
 \end{array}$$

PROP.

P R O P. II.

316. The Expression for the Superficies of a Sphere passing through the middle of each side of an Icofaedron, in the Terms of the circumscribing Sphere, is $R^2 e \times \frac{16S^2 - 3b}{4S^2}$.

For	{	1	$4 R^2$	} = Sq. of the Diam.
		2	$R^2 \times \frac{16S^2 - 3b}{4S^2}$	
				circumscr. 109
			of the Sphere	passing, &c. 312
		3	$4 R^2 e = \text{Sup. of circumsf. Sphere}$	103
But	{	4	$4 R^2 : R^2 \times \frac{16S^2 - 3b}{4S^2} :: 4 R^2 e$	
			$: R^2 e \times \frac{16S^2 - 3b}{4S^2} = \text{Superf.}$	106

The last otherways :

1 x e	{	1	$R^2 \times \frac{16S^2 - 3b}{4S^2}$	Square of Diameter of this Sphere	312
		2	$R^2 e \times \frac{16S^2 - 3b}{4S^2}$	Superficies	103

T H E O R E M.

Multiply a fourth part of the Superficies of the circumscribing Sphere, into 16 times the Square of the Sine of 36 Degrees made less by 3b; divide that Product by four times the Square of the Sine of

of 36 Degrees, and the Quotient will be the Superficies of this Sphere.

P R O P. III.

317. The Expression for the Solidity of a Sphere passing through the middle of each side of an Icosaedron, in the Terms of the circumscribing Sphere, is

$$\frac{R^3 e}{48 S^3} \times \overline{16 S^2 - 3 b}^{\frac{3}{2}}.$$

For	{	1	$8 R^3 = \text{Cube of the Diam.}$	}	of the
		2	$\frac{4 R^2 e}{3} = \text{Solidity}$		
		} circumscr. Sphere {			
					109
					101
But	{	3	$\frac{R^3}{8 S^3} \times \overline{16 S^2 - 3 b}^{\frac{3}{2}} = \text{Cube of the Diameter of this Sphere}$		312
		4	$8 R^3 : \frac{4 R^3 e}{3} :: \frac{R^3}{8 S^3} \times \overline{16 S^2 - 3 b}^{\frac{3}{2}} :$ $\frac{R^3 e}{48 S^3} \times \overline{16 S^2 - 3 b}^{\frac{3}{2}} = \text{Sol. \& c.}$		105

T H E O R E M.

Multiply a fourth part of the Solidity of the circumscribing Sphere, into the Square Root of the Cube of 16 times the Square of the Sine of 36 Degrees, made less by $3b$; and divide that Product by 16 times the Cube of the same Sine, and the Quotient will be the Solidity of this Sphere.

E L E



ELEMENTS

O F

Solid GEOMETRY

Algebraically Demonstrated.

B O O K III.

Containing Several curious

THEOREMS,

O U T O F

Archimedes's TREATISE

O F T H E

SPHERE and CYLINDER.

DEFINITIONS.



E T there be a Circle $BECG$, whose Fig. 20.
Center is A , its Diameter BC , which
let the Right-line EG cut at Right-
angles, but not through the Center
in D . Let there be drawn from the
Center, the Radius's AE , AG , this being sup-
posed, *Note*;

X

318. That

318. That a *Sector* of a Sphere is that which is produced from the *Sector* of a Circle $AECG$, or $AEBG$, turned round about the Diameter BC .

319. That the Segment of a Sphere is that Part or Portion of it, which is produced from the Segment of a Circle ECG , or EBG , turned round about the same Diameter BC .

320. The Vertex or Top of the Spherical Portion EBG , is the Extremity B , of the unmoved Diameter. The Base is the Circle described by the Line EG . The Axis that Part of the Diameter BD , which is intercepted between the Top B , and D , the Center of the Base.

321. When I name the Superficies of a Spherical Portion, or of a Body inscribed in it, or of a Cone, I always understand it without the Base; and when I say the Superficies of a *Cylinder*, I always mean without the Ends or Bases, unless the Word (whole) be adjoined to (Superficies;) for then the Bases are also meant to be taken in.

322. Again, when I speak of *Cylinders*, and *Cones*, I speak of no other but Right ones.

A X I O M S.

323. The Circuit of a *Polygon* inscribed in a Circle, is less than the Circumference of the Circle.

324. The Circuit of a *Polygon* described about a Circle, is greater than the Circumference of the Circle.

C O R O L L A R Y.

325. Hence it is plain, that as the number of sides is increased, the circumscribed and inscribed *Polygons* will approach nearer and nearer to the Circle;

Circle; and when the Number of Sides is infinite, the *Polygon* will be equal to, or end in the Circle, and also in one another.

326. And if a *Polygon* inscribed in a Circle be Fig. 21. turned round about the Diameter AE , together with the Circle, the Superficies of the Body produced by the *Polygon*, will be less than the Superficies of the *Sphere*; and if a *Polygon* circumscribed about a Circle, be turned about the Diameter AE , together with the Circle, the Superficies of the Body produced by the *Polygon*, will be greater than the Superficies of the *Sphere*.

COROLLARY.

327. Hence, and from *Art.* 325, if the Sides of the circumscribing and inscribed *Polygons* be infinitely many, the Bodies produced by their Rotation about the Diameter of the Sphere, will end in, and be equal to the Sphere, both in Superficies and Solidity.

328. In like manner, the Circuit of a *Polygon* Fig. 21. inscribed in the Segment of a Circle DAF , is less than the Circumference of the Segment; and if a *Polygon* inscribed in the Segment, be together with the Segment AO , turned round, the Superficies of the Body produced by the *Polygon*, will be less than the Superficies of the Spherical Portion DAF .

329. The Superficies of a *Prism* inscribed in a *Cylinder*, is less than the Superficies of the *Cylinder*; but the Superficies of the *Prism* circumscribed about the *Cylinder*, is greater than the Superficies of the *Cylinder*.

COROLLARY.

330. Hence if the Number of the sides of the *Prism* be infinite, it will end in, and be equal to the *Cylinder*.

331. And the Superficies of a *Pyramid* inscribed within a *Cone*, is less than the *Cone's* Superficies: But the Superficies of the circumscribed *Pyramid*, is greater than the Superficies of the *Cone*.

COROLLARY.

332. Hence if the polygonal Base of the inscribed and circumscribing *Pyramids*, have their Number of sides infinite, they will both end in, and be equal to the *Cone*.

P R O P. I.

Fig. 19. 333. A Regular *Polygon* $FINTR$, circumscribed about a Circle, is equal to a Triangle, whose Base is the Circuit of the *Polygon*, and its Height the Radius of the Circle.

And a regular *Polygon* inscribed within a Circle, is equal to a Triangle which hath for its Base the Circuit of the *Polygon*, and for its Height the Perpendicular AO .

Fig. 27. *Part I.* Place in a Right-line all the Sides of the *Polygon*, viz. BF, FI, IN, NI , &c. and they will compose the Right-line $BFINT R$: and at Right-angles thereunto, draw AB , equal to the Radius of the Circle: Then will the Triangles FAT, TAN , &c. be equal to one another (1. 6.) and each equal to a Triangle of the circumscribing *Polygon*, and therefore the whole Triangle RAB , is equal to the whole *Polygon*.

Part II. This is Demonstrated after the same manner as the first *Part* was.

COROL

COROLLARIES.

334. Hence the *Rule* for finding the Superficies of a *Polygon*, in *Art.* 29, is demonstrated : For if a Triangle, whose Base is equal to the Circuit of the *Polygon*, and Perpendicular equal to the Radius of its inscribed Circle, be equal to the *Polygon*, as it is by the last, and if the Superficies of a Triangle be found by multiplying the Perpendicular into half the Base, as in *Art.* 28, it must follow, that the Superficies of a *Polygon* is found by multiplying half the Circuit of the *Polygon* into a Perpendicular from the Center, or middle of the *Polygon*, to the middle of a Side, which is the Radius of the inscribed Circle.

335. Hence also the Reason for finding the Superficies of a Circle is evident : For if the *Polygon* have an infinite Number of Sides, it will end in the Circle, and so the Circuit of the *Polygon* will become the Circumference of the Circle, *per Art.* 325, and therefore half the Circuit of the *Polygon*, *i. e.* half the Circumference of the Circle, multiplied into the Radius, will give the Superficies of the Circle.

336. It is also hence evident, that a Rectangle under the Radius and whole Circumference, is double the Superficies of the Circle ; and a Rectangle under the Diameter and Circumference, is quadruple the said Superficies.

P R O P.

P R O P. II.

Fig. 29.

337. A Circle is to an inscribed Square, as half the Circumference $C F E$, is to the Diameter ; but to a Square circumscribed, as the fourth Part of the Circumference to the Diameter.

For the Rectangle under $C F E$, and the Radius CO , or OF , *i. e.* *per* last, the whole Circle is to the Rectangle $H L C E$, *i. e.* the Rectangle under $H L$, and $H C$, (*i. e.*) to the inscribed Square $B C F E$, as $C F E$, half the Circumference, is to $H L$, or $C E$ the Diameter, which was the first Thing : And consequently the Circle is to the double Rectangle $H L$, $C E$, (*i. e.* to $K H$, the circumscribing Square) as $C F E$ is to the double of the Diameter $C E$, or as the Quadrant $C F$, is to the Diameter $C E$.

P R O P. III.

338. The Circumferences of Circles, have the same Proportion between themselves, which their Diameters have.

For the Circuits of *Polygons* which may be inscribed in Circles without end, are always between themselves, as the Diameters of their circumscribing Circles, (*per* Cor. 1. 12.) but the Circuits end at length in the Circumferences, (*per* Art. 325.) therefore their Circumferences also are between themselves as their Diameters.

S C H O

SCHOLIUM.

339. From this is demonstrated the *Rule* in (*Art.* 30.) for finding the Circumferences of Circles whose Diameters are given; or the Diameters, having the Circumferences given.

For if the Circumference of a Circle whose Diameter is (1) be called (*e*.) and the Diameter of another Circle be called $2R$, it will be,

As $1 : e :: 2R : 2Re = \text{Circumference.}$ And

$$\text{As } e : 1 :: 2Re : \frac{2Re}{e} = 2R = \text{Diameter.}$$

And so (*e*) is the constant Ratio between the Diameter of a Circle and its Circumference.

P R O P. IV.

340. The Superficies of a *Prism*, as well that which is inscribed within, as that which is circumscribed about a *Cylinder*, is equal to a Rectangle, whose Height is the Side of the *Cylinder*, but its Base equal to the Circuit of the *Prism*.

For it is plain from *Art.* 35. that the Superficies of a *Prism*, is composed of as many Plains as the polygonal Base or End hath Sides, whose common Height is the Height of the *Prism*; therefore the Sum of all these Plains, that is the Circuit of the Base multiplied into the Height, will be equal to the Sum of all the Superficies's, or Superficies of all the Plains; *i. e.* equal to the Superficies of the *Prism*. And this is so whether a *Prism* be inscribed

bed within, or circumscribed about a *Cylinder*; for in either Case the Height of the *Prism* is the same with the Height of the *Cylinder*.

P R O P. V.

341. The Superficies of a regular *Pyramid* circumscribed about a right *Cone*, is equal to a Triangle, which hath for its Base the Circuit of the *Pyramid's* Base, but its Height the Side of the *Cone*.

342. And the Superficies of a regular *Pyramid*, inscribed within a *Cone*, is equal to a Triangle, which hath for its Base the Circuit of the pyramidical Base, but for its Height the Perpendicular let down from the Top into a Side of the Base.

Part I. For the *Pyramid* is composed of as many triangular Plains, as its polygonal Base hath Sides, *per Art. 9.* whose Bases are the Sides of the polygonal Base of the *Pyramid*, and Height the Side of the *Cone*; and therefore the Superficies of the *Pyramid*, is equal to the Sum of all those Superficies's, *i. e.* equal to the flant Height of the *Cone*, multiplied into the Height of the *Pyramid's* Base.

Part II. This *Pyramid* is contained under as many triangular Plains as the Base of the *Pyramid* hath sides, (as the last, and all other *Pyramids* are,) and the Sum of the Superficies's of all those Triangles, is found by multiplying the Perpendicular into half the Circuit, *i. e.* equal to the Triangle, &c.

P R O P.

P R O P. VI.

343. The Superficies of a *Prism* circumscribed about a right *Cylinder*, ends in the Superficies of the *Cylinder*; and the Superficies of a *Pyramid* circumscribed about a *Cone*, ends in the Superficies of the *Cone*.

For	{	1		ANB	}	= Superf. of the	{	$Prism$	38
									2
But	{	3		$NB = 2 R e$ when B is infinitely little					
$3 \times A$									4

The like Demonstration holds for the *Pyramid* and *Cone*.

P R O P. VII.

344. A Circle whose Radius is a mean Proportional, between the side of a right *Cylinder* and the Diameter of its Base, is equal to the *Cylindrical* Superficies.

For	1		$2 A R e =$ Superf. of the <i>Cylinder</i>	42
Put	2		$r =$ Radius of a Circle, = Superficies of the <i>Cylinder</i> .	
Then	3		$r^2 e =$ Superficies of the Circle, (per Art. 34) = $2 A R e$, per Hypoth.	
$3 \div e$	4		$r^2 = 2 A R$.	
4 Anal.	5		$A : r :: r : 2 R =$ Diam. of the <i>Cylinder</i> .	

Y

COROL.

COROLLARIES.

345. The Superficies of a right *Cylinder*, is equal to a Rectangle under the side of the *Cylinder*, and Circumference of its Base.

This is evident from *Art.* 42.

346. *Cylinders* of the same Height, have their Superficies one to another as the Diameters of their Bases.

$$\begin{array}{l} \text{For} \\ \text{But} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} 2 A R e = \text{Superficies of the Cyl.} \quad 42 \\ 2 A r e = 2 a r e \text{ (because } A = a, \text{ per} \\ \text{Hyp.)} = \text{Superf. of 'tother Cylinder} \\ 2 A R e : 2 A r e :: 2 R : 2 r, \text{ i. e.} \\ \text{as their Diameters.} \end{array}$$

347. The Superficies of *Cylinders*, that have equal Bases, are one to another as their Altitudes or Lengths.

$$\begin{array}{l} \text{For} \\ \text{But} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} 2 A R e = \text{Sup. of one Cylind.} \\ 2 a r e = 2 a R e, \text{ (per Hyp.)} = \\ \text{Sup. of the other Cylinder.} \\ 2 A R e : 2 a R e :: A : a, \text{ (i. e. as} \\ \text{their Altitudes, per 16.6} \end{array} \right\} \begin{array}{l} \\ 112 \\ \end{array}$$

348. Like *Cylindrical* Superficies have between themselves a duplicate Proportion of that of their Diameters of their Bases.

349. This

349. This Proposition is evident from *Art.* 106. but for Variety Sake we will demonstrate it as followeth.

$$\begin{array}{l} \text{For} \\ 1 \times 2 \end{array} \left\{ \begin{array}{l} 1 \mid A : a :: 2R : 2r, \text{ per Hypoth.} \\ 2 \mid 2Re : 2re :: 2R : 2r, \text{ per 16.6.} \\ 3 \mid 2ARE : 2are :: 4R^2 : 4r^2. \end{array} \right.$$

350. *Cylindrical* Superficies, have between themselves a Proportion compounded of the Proportion of the Sides and Diameters of their Bases.

$$\begin{array}{l} \text{For} \\ 1 \times 2 \\ 3 \div 2ARE \end{array} \left\{ \begin{array}{l} 1 \mid 2ARE : 2Are :: 2R : 2r \\ 2 \mid 2ARE : 2aRe :: A : a \\ 3 \mid 4A^2R^2e^2 : 4AaRre^2a :: 2AR : 2ar. \\ 4 \mid 2ARE : 2are :: 2AR : 2ar. \end{array} \right\} \begin{array}{l} \text{per Art. 346.} \\ \text{347.} \\ \left. \begin{array}{l} \text{when they} \\ \text{have} = \end{array} \right\} \begin{array}{l} \text{Altitudes} \\ \text{Bases} \end{array} \end{array}$$

351. If *Cylindrical* Superficies be equal, the Diameters will be reciprocally as their Altitudes.

$$\begin{array}{l} \text{For} \\ \text{But} \end{array} \left\{ \begin{array}{l} 1 \mid 2ARE = 2are, \text{ per Hyp.} \\ 2 \mid A : a :: 2r : 2R, \text{ per 16.6.} \end{array} \right.$$

SCHOLIUM.

352. By comparing these *Corollaries* with the *Propositions* in the First Book concerning the *Cylinder*, it will easily appear that the same Proportion takes place with Respect to Superficies, as it does with Respect to Solidity, in corresponding Cases; and it is evident that the same Observation holds good in the *Prism*, *Cone*, and *Pyramid*.

P R O P. VIII.

353. The Superficies of a right *Cylinder*, is to the Base, as the side of the *Cylinder*, is to a fourth Part of the Diameter of the Base.

$$\begin{array}{l} \text{For} \quad \left\{ \begin{array}{l} 1 \mid 2 \ A R e = \\ 2 \mid \quad R^2 e = \end{array} \right\} \begin{array}{l} \text{Superf. of the } \{ \text{Cylinder} \\ \text{Base} \end{array} \quad \begin{array}{l} 42 \\ 31 \end{array} \\ \text{But} \quad \left| 3 \mid 2 \ A R e : R^2 e :: A : \frac{2 R}{4} \left(= \frac{R}{2} \right) \right. \quad \begin{array}{l} p. 16.6 \end{array} \end{array}$$

C O R O L L A R I E S.

354. The Superficies of a right *Cylinder*, which hath its side equal to the Diameter of its Base, is fourfold of its Base.

$$\begin{array}{l} \text{For} \quad \left\{ \begin{array}{l} 1 \mid 2 \ A R e = 4 R^2 e \text{ (because } A = 2 R) \\ \quad \text{per Hyp.} = \text{Cylinder} \end{array} \right\} \quad \begin{array}{l} 42 \\ 31 \end{array} \\ \text{But} \quad \left| 3 \mid 4 R^2 e : R^2 e :: 4 : 1, \text{ or fourfold,} \right. \quad \begin{array}{l} p. 16.6 \\ 355. \text{ If} \end{array} \end{array}$$

355. If the side of a *Cylinder* be equal to a fourth Part of the Diameter of the Base, the Superficies of the *Cylinder* will be equal to the Superficies of the Base.

For	{	1		2ARE = R ² e (because A = $\frac{2R}{4} = \frac{R}{2}$)	42
But	{	2		R ² e = Superficies of the Base	31
				3	

P R O P. IX.

356. A Circle, whose Radius is a mean Proportional between the side of a right *Cone*, and the Radius of its Base, is equal to the Conical Superficies: For the side of the *Cone* put S.

For		1		SR e = Superf. of the Cone	45
Put				2	
Then		3		r ² e = Superf. of the Circle (per Art. 34.) = SR e, per Hyp.	31
3 ∴				4	

C O R O L L A R I E S.

357. The Superficies of a right *Cone*, is equal to a Triangle comprehended under the side of the *Cone* and the Circumference of the Base.

This

This is plain from *Art.* 325 and 333.

358. The *Conical* Superficies having their sides equal, are betwixt themselves as the Diameters of their Bases.

359. Those *Cones* which have equal Bases, have their Superficies in Proportion as their sides.

360. Those *Conical* Superficies, which are like, have between themselves a Proportion duplicate to that which is between the Diameters of their Bases.

361. All *Conical* Superficies whatsoever, have between themselves a Proportion, which is compounded of the sides and Diameters of their Bases.

362. Those *Conical* Superficies which are equal, have their sides, and Diameters of their Bases reciprocally proportional; and those which have them so are equal.

All these are demonstrated as their corresponding *Corollaries* in *Proposition* VII.

P R O P. X.

363. The Superficies of a right *Cone*, is to the Superficies of the Base, as the side is to the Radius of the Base.

$$\text{For } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. \left| \begin{array}{l} A R e = \text{Superficies of the Cone.} \\ R^2 e = \text{Superficies of the Base.} \\ A R e : R^2 e :: A : R, \text{ per 16.6.} \end{array} \right. \begin{array}{l} 45 \\ 31 \\ \end{array}$$

COROL.

COROLLARIES.

364. The Superficies of a right *Cone* produced Fig. 20. by an Equilateral Triangle turned round about the Perpendicular *BD*, is double to the Base.

$$\begin{array}{lcl} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} ARe (= \text{Superficies of the Cone}) \\ = 2R^2e, \text{ per Hyp.} \end{array} \right. & \begin{array}{l} 45 \\ 31 \end{array} \\ 1, 2 & \left| \begin{array}{l} 2 \\ 3 \end{array} \right. & \begin{array}{l} R^2e = \text{Superf. of the Base.} \\ 2R^2e : R^2e :: 2 : 1, \text{ per 16.6} \end{array} \end{array}$$

365. The Superficies of a *Cone* produced by a Fig 25. Right-angled Equicrural Triangle, is to the Superficies of the Base, as in a Square *EDBQ* the Diameter is to the side.

$$\begin{array}{lcl} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} ARe = R^2e\sqrt{2} \ (A = R\sqrt{2}) \text{ Superfi-} \\ \text{cies of the Cone } EBD \end{array} \right. & \begin{array}{l} 45 \\ 31 \end{array} \\ \text{But } \left\{ \begin{array}{l} 2 \\ 3 \end{array} \right. & \left| \begin{array}{l} R^2e = \text{Superficies of the Base} \\ R^2e\sqrt{2} : R^2e :: R\sqrt{2} \ (EB) : R \\ \quad (EA) \text{ per 16.6} \end{array} \right. & \end{array}$$

366. The Superficies of a right *Cylinder*, is to the Superficies of a right *Cone* of equal Base and Altitude with it, as the side of the *Cylinder*, is to half the side of the *Cone*.

For

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left. \begin{array}{l} 2 \text{ } A R e \\ A R e \end{array} \right\} = \text{Super.} \left\{ \begin{array}{l} \text{Cylinder} \\ \text{of the Cone} \end{array} \right. \quad \begin{array}{l} 42 \\ 45 \end{array} \\ \text{But} & \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. & \left. \begin{array}{l} 2 \text{ } A R e : A R e :: A : \frac{1}{2} A, \text{ per 16.6} \end{array} \right. \end{array}$$

P R O P. XI.

Fig. 26. 367. If a right *Cone* be cut by a Plane $Q R S$, parallel to the Base $N O Z$, I say that the Circle, whose Radius is a mean Proportional between Part of the side $N Q$, and $Q D$, $N V$, the Radius's of the Circles $Q R S$, $N Z O$ taken together, is equal to the Conic Superficies intercepted between the parallel Circles $Q R S$, $N Z O$.

Make $R = N V$, $r = Q D$, and $S = N P$.

$$\begin{array}{lcl} \text{For} & 1 & R : r :: S : \frac{S r}{R} = N Q, \text{ per 4.6.} \\ 1 \text{ } S - \frac{S R}{R} & 2 & \frac{R S - S r}{R} = N Q, \text{ per Fig.} \\ & 3 & \left. \begin{array}{l} \frac{r^2 e S}{R} \\ R e S \end{array} \right\} = \text{Superficies} \left\{ \begin{array}{l} P Q B \\ \text{of the Cone} \\ N P O \end{array} \right\} \quad 45 \\ & 4 & \\ 4 - 3 & 5 & \frac{R^2 e S - r^2 e S}{R} = \text{Superf. } N Q B O. \\ \text{Suppose} & 6 & d = \text{Radius of the Circle proposed.} \\ \text{Then} & 7 & d^2 e = \text{Superficies of the Circle.} \\ & 8 & \frac{R^2 e S - r^2 e S}{R} = d^2 e, \text{ per Hyp.} \\ \text{And} & & \\ 8 \div e \text{ Anal.} & 9 & \frac{R S - r S}{R} (N Q) : d :: d : R + \\ & & (= Q D + N V) \text{ per 16.6.} \end{array}$$

L E M

L E M M A.

368. Right-lines HB , GC , which in a Circle in-Fig. 18. intercept equal Arches BC , GH , are parallel.

For let CH be drawn; because the Arches BC , HG , are equal, per Hypotheses, the Alternate Angles also BHC , GCH , are equal, per 29.3; therefore HB , GC are parallel, per 28.1.

P R O P. XII.

369. Let there be inscribed in a Circle a regu-Fig. 18. lar Figure of an even Number of sides, and let it be equilateral; let EB be drawn from the Extremity of the Diameter, unto B , the End of the side next to the Diameter, and let the right Lines BH , CG , DF , join the Angles which are equally distant from A .

I say, that the Rectangle contained under the Diameter AE , and the Subtense EB , is equal to the Rectangle contained under one side of the inscribed Figure AB , or BC . and of all the joining Lines HB , CG , DF , taken together.

Draw CH and DG ; because HB , CG , DF , intercept equal Arches BC , GH , &c. these Lines will be parallel, (per *Art.* 368.) by the same Argument BA , CA , DG , EF , are parallel; all the Triangles therefore BAK , KHL , LCM , MGN , NDO , OFE , are equiangular, (per 27. and 15.1) therefore if $BK = KH = DO = OF$, be called x ,
Z
and

and $KA.a$, $KL.b$, $ML.c$, $MN.d$, $NO.e$, and $OE.f$, and $CM = MG.z$, then we shall

have	1	$x : a :: x : b :: z : c :: z : d :: x : e$ $:: x : f$, per 4.6.
But	{	2 $x : a :: 4x + 2Z (= BH + CG + DF)$ $: a + b + c + d + e + f (EA)$ p. 12.5
		3 $x : a :: BE : BA$, per 8.6.
2, 3	4	$BH + CG + DF : EA :: BE : BA$, per 11.5.
4 ∴	5	$BA \times BH + CG + DF = EA + BE$.

P R O P. XIII.

Fig. 18. 370. Let there be inscribed in DAF a Segment of a Circle, whose Base DF is perpendicular to the Diameter AOE a Figure equilateral, and of an even Number of sides, and let there be drawn as in the Figure foregoing the Line EB .

I say, that the Rectangle comprehended under EB , and AO , part of the Diameter, is equal to the Rectangle which is comprehended under one side of the inscribed Figure, and all the joining Lines BH , CG , &c. taken together, with DO half the Base.

Suppose

Suppose every thing as in the last.

For	{	1 $x : a :: x : b :: Z : e :: Z : d :: x$ $: e, \text{ per } 4.6.$ 2 $x : a :: 3x + 2Z (= BH + GC + DO) : a + b + c + d + e (= AO)$ $(\text{per. } 12.5)$
But	{	3 $x : a :: BE : AB, \text{ per } 8.6.$ 4 $BH + GC + DO : AO :: BE : AB.$ 5 $AB \times \overline{BH + GC + DO} = AO \times BE.$

L E M M A.

371. Let there be inscribed in the greatest Circle of the Sphere, a regular Figure which hath its sides measured by the Number Four, and which stands about the *Axis* AE , which *Axis* remaining unmoved, let the Circle be turned about, together with the Figure. Fig. 18.

I say that there will be inscribed in the Sphere a Body, contained under *Conical* Superficies.

It is manifest that BA , HA , likewise DE , FE , describe entire Surfaces of right *Cones*; then because the Lines CB , GH , &c. GI , CD , being produced do concur on both sides the point in the Diameter AE , which is in like manner to be drawn out, and cuts the joining Lines perpendicularly, it is also manifest that the said Lines CB , GH , &c. do describe parts of right *Cones*, which are intercepted between parallel Circles, which the Tops of the Angles B , C , D , design in the Spherical Superficies.

Z 2

Let

Let DAF be the greatest Segment of a Sphere whose Axis is AO ; Let there be inscribed in this a Figure having all the sides equal, the Base excepted; let it be turned round about the Axis; I say that a Body contained under a Conical Superficies will be inscribed in the same Segment.

This is demonstrated as the foregoing Lemma.

P R O P. XIV.

Fig. 18. 372. Let the same things be supposed which were in the first Lemma, and let the right Line EB be drawn, from the Extremity of the Diameter unto the end of the side next to the Diameter.

I say that a Circle, whose Radius (I) is a mean Proportional between the Diameter AE , and the Subtense BE , is equal to all the Conical Superficies inscribed in the Sphere.

Let every thing be supposed as in Art. 369.

For	{	1	$ABe \times x =$	} Superficies of
		2	$ABe \times x + Z =$	
			$\{ABH$	} 45
			$\{CGHB$	
$1 + 2 \times 2$		3	$ABe \times 4x + 2Z (= \text{all the Conical Superficies}) = AE \times BE \times e$	369
Put		4	$I = \text{Radius of the Circle proposed}$	
Then		5	$I^2 = \text{Superficies of the Circle.}$	
$3 = 5$		6	$AE \times BE \times e = I^2 e$, per Hyp.	
$6 \div e$ Anal.		7	$AE : I :: I : BE$, per 16.6.	

P R O P.

P R O P. XV.

373. Let the same things be supposed which Fig. 18. were in *Art.* 371, and let the right Line EB be drawn from the Extremity of the Diameter AE , to the end of AB , the side next to the Diameter.

I say that a Circle, whose Radius is a mean Proportional between EB , and AO , the Axis of the Spherical Portion, is equal to all the *Conical* Superficies inscribed in the Spherical Segment.

Let all be as before,

For	{	1	$ABe \times x$	} = Superficies of	
		2	$ABe \times \frac{x}{x+Z}$		
			the Cone	$\left\{ \begin{array}{l} BAH \\ CGHB \end{array} \right\}$	45
$2 \times \frac{1}{2} + 1$		3	$ABe \times 3x + 2Z$	(= all the Superficies) = $BE \times AO \times e$.	
Put		4	I = Radius of the proposed Circle		
Then		5	$I^2 e$ = Superficies of the Circle		
$3 = 5$		6	$ABe \times 3x + 2Z = BE \times AO \times e$ = $I^2 e$, per Hyp.		
$6 \div e$ Anal.		7	$AO : I :: I : BE$.		

P R O P. XVI.

374. Every Sphere is quadruple of the *Cone*, whose Base is equal to the greatest Circle of the Sphere, but its Altitude equal to the Radius of the Sphere.

For

For $2, \frac{R}{3}$ $1, 3$	{	1	$\frac{4R^3e}{3} = \text{Solidity of a Sphere. } 101$
		2	$R^2e = \text{Base of the Cone per Hyp. } 31$
		3	$\frac{R^3e}{3} = \text{Solid. of the Cone.}$
		4	$\frac{4R^3e}{3} : \frac{R^3e}{3} :: 4 : 1, \text{ per 16.6.}$

P R O P. XVII.

375. A Sphere whose Radius is a mean Proportional between the Radius of a Cone's Base, and its Altitude, will be in Proportion to the Cone, as four times the Radius of the Sphere, is to the Radius of the Cone's Base.

For the Radius of the Cone's Base put r .

Then $2, 3$ But	{	1	$r : R :: R : \frac{R^2}{r} = \text{Altitude of the Cone, per Hyp.}$
		2	$\frac{4R^3e}{3}$
		3	$\frac{R^2re}{3}$
		4	$\frac{4R^3e}{3} : \frac{R^2re}{3} :: 4R : r, \text{ per Hyp.}$

}

Solid. of the

{

Sphere 101

Cone 97

P R O P.

P R O P. XVIII.

376. *Conical* Superficies inscribed in a Sphere, Fig. 21.
do at length end in the Superficies of the Sphere.

For	1	$AEe \times BE = AB \times e \times \overline{4x + 2Z} =$	
		<i>Conical Superf.</i>	45
But	2	$AE = BE$, when AB is infinitely little	
1, 2	3	$\overline{AE^2} \times e (= ABe \times \overline{4x + 2Z}) =$	
		Area of the Sphere	103

P R O P. XIX.

377. It was demonftrated (*Art.* 372.) that a Cir- Fig. 21.
cle whose Radius is a mean Proportional betwixt
the Diameter AE , and the right Line EB , which is
drawn from the Extremity of the Diameter unto
the End of the fide AB next to the Diameter, is
equal to all the *Conical* Superficies inscribed in the
Sphere.

I fay that this Circle ends at length in a Circle
whose Radius is AE the Diameter of the Sphere.

Let I ftill be the Radius of the Circle as in
Art. 372.

For	1	$AE \times BE = I^2.$	372
But	2	$AE = BE$, when AB is infinitely little	
1, 2	3	$AE^2 = I^2$ or $AE = I.$	

P R O P.

P R O P. XX.

Fig. 21. 378. It was demonstrated at *Art.* 373, that a Circle whose Radius was a mean Proportional betwixt BE and the Axis of a Segment AO , is equal to all the *Conical* Superficies inscribed in the same Segment DAF .

I say that this Circle ends at length in a Circle, whose Radius is the right Line AD , drawn from the Vertex of the Segment, unto the Circumference of the Circle $DQFN$, which is the Base of the Segment.

(I) being Radius as before,

For	1	$BE \times AO = I^2.$	273
But	2	$BE = AE$, when AB is infinitely little.	
Also	3	$AE (BE) : AD :: AD : AO$, per <i>Cor.</i> 8.6.	
1, 2, 3	4	$AE \times AO = AD^2 = I^2$, i. e. $AD = I.$	

P R O P. XXI.

379. The Superficies of every Sphere, is four-fold of the Superficies of the greatest Circle of the Sphere.

For	1	$R^2 e = \text{Superf. of the greatest Circle}$	31
But	2	$4 R^2 e = \text{Superf. of the Sphere}$	103
	3	$4 R^2 e : R^2 e :: 4 : 1$, per 16.6	

COROL.

C O R O L L A R Y.

380. From this admirable *Theorem*, whereby *Archimedes* hath purchased to himself an immortal Name, among the *Geometricians* of the first Rank, a Circle equal to a Spherical Superficies is obtained, that, to wit, whose Semi-diameter is the Diameter of a Sphere, or whose Diameter is double to the Sphere's Diameter.

For if $2R$ (which is the Diameter of the *Sphere*) be made Radius of a Circle,

Then $4R^2e = \text{Superficies of the Circle (per Art. 31.)} = \text{Superficies of the Sphere.}$ 103

S C H O L I U M.

381. We are now well provided for measuring of a Spherical Superficies, the chief of all the *Geometrical Curves*; and it is performed these two ways.

First, let the greatest *Circle* of the Sphere be measured, per *Art. 31.* and multiply that by 4, and the Product will be the Superficies of the Sphere.

382. Secondly, the Diameter of the Sphere multiplied into the Circumference of the greatest *Circle*, gives you the Spherical Superficies.

A a

P R O P.

P R O P. XXII.

Fig. 21. 383. The Superficies of any Spherical Portion whatever, as DAF , is equal to a *Circle* whose Radius is the Right-line AD , drawn from the Vertex of the Portion to the Circumference of the *Circle* $DFIN$; which is the Base of the Segment.

This is only *Art.* 378. in other Words, and therefore needs no Demonstration.

P R O P. XXIII.

Fig. 23. 384. The Superficies of a right *Cylinder* circumscribed about a *Sphere*, as the *Cylinder* $HPSV$, is equal to the Superficies of the *Sphere*.

385. And if the *Cylinder* and *Sphere* be both cut by a plain perpendicular to the *Axis* BG , each Segment of the *Cylinder's* Superficies will be equal to each corresponding Segment of the *Sphere*.

Part I. Let the Diameter of the *Sphere* $2R$, be equal to the Height of the *Cylinder*, as per *Hypothesis*.

Then, 1. $4R^2e$ = Superficies of the Square, (per *Art.* 103) = Superficies of the *Cylinder*, per *Art.* 42.

2. Draw the Lines BO , GO , then will the Angle BOG be a right Angle, per 31.3; and draw CO perpendicular to BG .

For	{	1	$\overline{BO}^2 = EC \times BG$ (per Cor. 2.8.6) = $IF \times HI$, per Fig.	
		2	$\overline{BO}^2 \times e$ (=Circle BO , per Art. 31.) = Segm. of the Cylinder HI	42
		3	$\overline{BO}^2 \times e$ = Segment of the Sphere BOK	383
		4	Cylinder's Segment HI = Sphere's Segment BOK	54

P R O P. XXIV.

386. The Segments of Spherical Superficies divided by parallel Circles, have that Proportion amongst themselves, which the Segments BC , CD , DA , AE , EF , FG , of the Diameter BG , which is perpendicular to the parallel Circles, have amongst themselves.

For	{	1	Sph. Seg. OBK : Sph. Seg. QBR :: Cyl. Seg. HT : Cyl. Seg. HX	385
		2	BC : BD :: Cyl. Segm. HT : Cyl. Segm. HX .	347
1, 2		3	Sph. Segm. OBR : Sph. Segm. QBR :: BC : BD , per 11.5.	
theref.		4	Sph. Seg. QBR — Sph. Seg. OBK (=Sph. Segm. OQR) : Sph. Segm. OBK :: DB — BC (= CB) : DC . per 17.5.	

SCHOLIUM.

387. From hence the Proportion of *Zones* and *Climates* between themselves become known, for they are one to another as the Segments of their Axis, *i. e.* as the Sine of Latitude.

388. From the same also we learn to measure the Segments of Spherical Surfaces; for, because both the whole Surfaces of the Sphere is known from *Art.* 103, and the Proportion of the Segments of the same, as that of the Parts of the Axis also given; it is manifest that each of the Segments become known.

Now both the foregoing, and all the rest of the *Theorems* which follow, are altogether singular and admirable, and well worth that those that are studious in *Geometry* should give great Diligence to understand them.

P R O P. XXV.

389. Every *Sphere* is equal to a *Cone*, whose Altitude is equal to the Radius of the *Sphere*, and Base equal to the Superficies of the *Sphere*.

For	1	$4 R^2 e = \text{Sph. Superf. (per Art. 103.)}$ $= \text{Cone's Base, per Hyp.}$
$1 \times \frac{R}{3}$	2	$\frac{4 R^3 e}{3} = \text{Sol. of the Sph. (Art. 101)}$ $= \text{Sol. Cone, per Art. 97.}$

P R O P.

P R O P. XXVI.

390. Every *Sector* of a *Sphere* is equal to the *Cone*, whose *Altitude* is the *Radius* of the *Sphere*, and *Base* the *Spherical Superficies* of the *Spherical Sector*.

This is very evident from *Art.* 26.

C O R O L L A R Y.

391. Seeing the *Superficies* ECG , is equal to Fig. 20.
a *Circle*, whose *Radius* is the *Right-line* CG , and the *Superficies* EBG is equal to a *Circle* whose *Radius* is BG , both (per *Art.* 383.) the *Sectors* $AEBG$ and $AECG$, will be equal to *Cones*, whose *Altitude* is the *Radius* of the *Sphere*, and their *Bases* *Circles*, whose *Radius's* are CG and BG .

S C H O L I U M.

392. From these Things is deduced the measu- Fig. 21.
ring both of *Sectors*, and *Segments* of *Spheres* of *Sectors*, as appears from (*Art.* 390); and if a third Part of the *Radius* be multiplied into the *Superficial Contents* of the *Sectors*, which is already known from (*Art.* 383.), or by the *Circle* whose *Radius* is CG or BG , and if *Segments* of the *Cone* EAG , be measured, and be taken away from the *Sector*, if it be less than an *Hemisphere*, but added thereunto if it be greater.

The *Segment* $MQRN$, which lies between Fig. 23.
two *Circles*, whether parallel or not parallel,
is

is measured, if the Segments QBR and MBN , already known, be subtracted one from another.

P R O P. XXVII.

Fig. 25. 393. An Hemisphere $EOBD$, is double to the Cone EBD , which hath the same Base and Altitude with it ſelf.

$$\begin{array}{lcl} \text{For} & \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left| \begin{array}{l} \frac{2 R^3 e}{3} = \text{Hemisphere } EOBD. \\ R^2 e \times \frac{R}{3} = \frac{R^3 e}{3} = \text{Solidity of the} \\ \text{Cone } EBD. \end{array} \right. \\ \text{But} & 3 & \left| \begin{array}{l} \frac{3 R^3 e}{3} : \frac{R^3 e}{3} :: 2 : 1, \text{ or double,} \\ \text{(per 16.6)} \end{array} \right. \end{array}$$

P R O P. XXVIII.

Fig. 28. 394. Let a *Sphere* be divided into two Segments $ILBG$, $ISKG$, by the Plain $IQGI$, which does not paſs through the Center A ; and let the Diameter BOD , be perpendicular to the cutting Plain.

As the Altitude OB of the Segment $ILBG$, is to the Radius of the *Sphere* AB , ſo let OK the Altitude of the other Segment, be made to the other Line KN .

395. In like manner, as OK the Altitude of the Segment $ISKG$, is to Radius AK , or AB ; ſo let the Altitude OB , of the other Segment, be made

made to the other Line BD ; which Things being supposed, I say,

396. The Cones ING , IDG , whose Altitudes are ON , OD , and $I\mathcal{Q}GT$, their common Base, are equal to their Spherical Segments.

There is the same Proportion of the Segments, as there is of the right Lines DO , NO .

397. The Segment $ISKG$, is to the greatest Cone IKG , inscribed in it, as NO to KO ; and the Segment ILB , is to the greatest Cone IBG inscribed in it, as DO is to BO .

Call the Diameter BK , $2R$; and OK call a .

For	{	1	$2 R e a$	} = Circle upon $\{IK\}$ $\{SO\}$ (per Cor. 2, 8, 6, and Art.
		2	$2 R e a - e a^2$	
$1 \times \frac{R}{3}$		3	$\frac{2 R^2 e a}{3}$	= Sol. of the Sector $AIKG$.
$2 \times \frac{R-a}{3}$		4	$\frac{2 R^2 e a - 3 R e a^2 + a^3 e}{3}$	= Cone IAG
But		5	$2 R - a : R :: a : \frac{R a}{2 R - a}$	= KN , (per Hyp.
$3-4$		6	$\frac{3 R e a^2 - a^3 e}{3}$	= Sol. of the Segm. Sphere $ISKG$.
$5+a$		7	$\frac{3 R a - a^2}{2 R - a}$	= ON , per Fig.
$7 \times 2 \& \div 3$		8	$\frac{3 R e a^2 - a^3 e}{3}$	= Solid. of the Cone ING (per Art. 97.) = Segm. Sphere $ISKG$, per Step 6th.

Part

Part II. This is plain from Part I.

For	{	1	Segm. of the Sphere $ISK G = \text{Cone } I N G$, per last.
		2	Segm. of the Sphere $ILB G = \text{Cone } ID G$, per last.
But		3	$\text{Cone } ID G : \text{Cone } I N G :: DO : NO$ (81 and 96)
Theref.		4	Segm. of the Sphere $ISB G : \text{Segm. } ISK G :: DO : NO$, per 1st, 2d, and 3d Steps.

Part III. This also is plain from Part I.

For	{	1	Segm. Sphere $ISK G = \text{Cone } I N G$ 396
		2	$\text{Cone } I N G : \text{Cone } I K G :: NO : OK$ (81 and 96)
But		3	Segm. of the Sphere $ISK G : \text{Cone } I K G :: NO : OK$, per 1st and 2d Steps.

SCHOLIUM.

398. From the first Part of this Proposition, there ariseth another way of measuring Spherical Segments, and that a very easy one, if, to wit, the Cones $ID G$, $I N G$, be measured, which will be done, if the third Part of the right Lines DO , NO , be multiplied into the Area of the Circle $Q T$.

P R O P.

P R O P. XXIX.

399. A right *Cylinder*, is both in Solidity, and whole Superficies, to the *Sphere* about which it is circumscribed, as 3 to 2.

Part I. For the Solidity.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \\ 2 \end{array} \middle| \begin{array}{l} 4 R^3 e \\ 3 \\ 2 R^3 e \end{array} \right\} = \text{Solid.} \left\{ \begin{array}{l} \text{Sphere} \\ \text{of the} \\ \text{Cylinder} \end{array} \right. \begin{array}{l} 101 \\ \\ 49 \end{array} \\ \text{But} \left\{ \begin{array}{l} 3 \end{array} \middle| \begin{array}{l} 2 R^3 e : \frac{4 R^3 e}{3} :: 3 : 2, \text{ p. 6. 6.} \end{array} \right\} \end{array}$$

400. *Part II.* For the whole Superficies.

$$\begin{array}{l} \text{For} \left\{ \begin{array}{l} 1 \\ 2 \end{array} \middle| \begin{array}{l} 4 R^2 e \\ 6 R^2 e \end{array} \right\} = \text{Superf.} \left\{ \begin{array}{l} \text{Sphere} \\ \text{of the} \\ \text{Cylinder} \end{array} \right. \begin{array}{l} 103. \\ \\ +2. \end{array} \\ \text{But} \left\{ \begin{array}{l} 3 \end{array} \middle| \begin{array}{l} 6 R^2 e : 4 R^2 e :: 6 : 4 :: 3 : 2, \text{ per} \\ (11.5) \end{array} \right\} \end{array}$$

SCHOLIUM.

It is an Argument what a value *Archimedes* put upon this *Theorem*, that he would have a *Sphere* inscribed in a *Cylinder*, cut upon his Tomb. And perhaps amongst so many other famous Discoveries, this chiefly, and above all others pleased him, for this Reason; to wit, because one and the same rational proportion takes place, both in the Bodies and in the Surfaces which contain them.

P R O P. XXX.

Fig. 22. 401. The Superficies of a Sphere hath that proportion to the whole Superficies of a square *Cylinder* inscribed in it, which 4 hath to 3.

Put $2R$ for AP the Diameter of the Sphere,

Then	1	$R\sqrt{2} = AK = PK$, per 47.1.	
$1 \times e$	2	$R e \sqrt{2} = \text{Circumference}$	} of the Circle
$2 \times \frac{R e}{4}$	3	$\frac{R^2 e}{2} = \text{Superficies}$	
1×2	4	$2 R^2 e = \text{Superficies}$	} of the
$3 \times 2 \& \frac{1}{4}$	5	$3 R^2 e = \text{whole Superf.}$	
But	6	$4 R^2 e = \text{Superficies of the Sphere.}$	
5, 6	7	$4 R^2 e : 3 R^2 e :: 4 : 3$, per 16.6.	

P R O P. XXXI.

402. The Superficies of a Sphere is double to the Superficies of a square *Cylinder* inscribed in the same *Sphere*.

For	{	1	$2 R^2 e$	= Superfi-	{	<i>Cylinder</i> . 42.
		2	$4 R^2 e$	cies of the		<i>Sphere</i> . 103.
But		3	$4 R^2 e : 2 R^2 e :: 2 : 1$			, per 16.6.

C O R O L.

COROLLARIES.

403. The whole Superficies of a right *Cylinder* described about a Sphere, is to the whole Superficies of an equilateral or Square *Cylinder* inscribed in it, as 2 to 1.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} 6R^2e : 4R^2e :: 3 : 2, \text{ p. 400.} \\ 3R^2e : 4R^2e :: 3 : 4, \text{ p. 401.} \end{array} \\ \text{But } \left| \begin{array}{l} 3 \end{array} \right| 6R^2e : 3R^2e :: 4 : 2 :: 2 : 1. \end{array}$$

404. The Superficies of a Sphere, is to the Superficies of a Square *Cylinder* inscribed in it, as the whole Superficies of a right *Cylinder* circumscribed about the Sphere, is to the whole Superficies of a right *Cylinder* inscribed within the same Sphere.

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} 4R^2e : 2R^2e :: 2 : 1. \\ 6R^2e : 3R^2e :: 2 : 1. \end{array} \\ \text{But } \left| \begin{array}{l} 3 \end{array} \right| 4R^2e : 2R^2e :: 6R^2e : 3R^2e \text{ per} \\ \hspace{15em} (11.5) \end{array}$$

P R O P. XXXII.

405. The Portion of any spherical Superficies Fig. 28. whatever, as *IL*, *BG*, hath the same Proportion to the Superficies of the greatest inscribed *Cone*, which *BG*, the side of the *Cone*, hath to *GO*, the Radius of the Base.

B b 2

For

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \mid \overline{BG}^2 \times e \\ 2 \mid GO \times e \times BG \end{array} \right\} & = \text{Superf. of the} & \left\{ \begin{array}{l} \text{Segm. of} \\ \text{Cone} \end{array} \right. \\
 & & \left\{ \begin{array}{l} \text{the Sphere } ILBG, \\ IBG, \end{array} \right\} \begin{array}{l} \text{per 383.} \\ \text{per 45.} \end{array} \\
 \cdot \quad 1, 2 \quad \left\{ \begin{array}{l} 3 \mid \overline{BG}^2 \times e : GO \times e \times BG : : \text{Seg. Sph.} \\ \quad \quad \quad ILBG : \text{Cone } IBG. \end{array} \right. & & \\
 3 \div BG e \quad \left\{ \begin{array}{l} 4 \mid BG : GO : : \text{Seg. Sph. } ILBG : \text{Cone} \\ \quad \quad \quad IBG. \end{array} \right. & &
 \end{array}$$

P R O P. XXXIII.

Fig. 25. 406. The Superficies of the Hemisphere $EOBD$, hath that Proportion to EBD , the Superficies of the greatest inscribed Cone, which in the Square $BADK$, the Diameter BD , hath to the side AD ; and the same Proportion to the Superficies of a right Cone ebd , circumscribed about it, as the side hath to the Diameter.

Part I.

$$\begin{array}{lcl}
 \text{For } \left\{ \begin{array}{l} 1 \mid \overline{BD}^2 \times e = \text{Superficies of the He-} \\ \quad \quad \quad \text{misphere of } EOBD \quad 383. \\ 2 \mid AD \times e \times BD = \text{Superficies of the Cone} \\ \quad \quad \quad EBD \quad 45. \end{array} \right\} & & \\
 1, 2 \quad \left\{ \begin{array}{l} 3 \mid \overline{BD}^2 \times e : AD \times e \times BD : : \text{Hemisph.} \\ \quad \quad \quad EOBD : \text{Cone } EBD. \end{array} \right. & & \\
 3 \div BD \times e \quad \left\{ \begin{array}{l} 4 \mid BD : AD : : \text{Hemisph. } EOBD : \\ \quad \quad \quad \text{Cone } EBD. \end{array} \right. & &
 \end{array}$$

407. Part II.

For	{	1	$\overline{BD}^2 \times e =$ Superficies of the Hemisphere $EOBD$. 383
		2	$Ad \times e \times bd =$ Superficies of the Cone EBD . 45
		3	$BD = Ad$, as is easily demonstrated
		4	$AD : BD :: Ad (= BD) :: \frac{\overline{BD}^2}{AD}$ (per 16.6) $= bd$.
2, 3, 4,		5	$\frac{\overline{BD}^3 \times e}{AD} =$ Superficies of the Cone ebd .
But		6	$\overline{BD}^2 \times e : \frac{\overline{BD}^3 \times e}{AD} :: EOBD : ebd$ $:: AD : BD$, per 16.6

C O R O L L A R Y.

408. The Superficies of the greatest right Cone EBD , inscribed in an Hemisphere $EOBD$, is in Proportion to the Superficies of a right Cone ebd circumscribed about it, as the Radius of the inscribed Cone's Base AD , is to bd the side of the circumscribed Cone.

For

For	{	1	Hemisphere $EOBD$: Cone EBD :: $BD : AD$. 406
		2	Hemisphere $EOBD$: Cone ebd :: $AD : BD :: Ad : bd :: BD : bd$.
But	}	3	Cone EBD : Cone ebd :: $AD : bd$.

P R O P. XXXIV.

409. A Sphere hath that Proportion to a Square conical *Rhombus* circumscribed about it, both in Respect of Superficies and Solidity, which in a Square the side hath to the Diameter.

Fig. 29. Let the Square $EBCF$ be circumscribed about $HGDI$ the greatest Circle of the *Sphere*, from which Square, as turned about the Axis BF , let a conical *Rhombus* encompassing the *Sphere* be produced.

Put $GI = FC = CB = \&c. = 2R$, then is,

Part I.

For	{	1	$2R\sqrt{2} = BF = EC$, per 47.1.
		2	$2Re \times \sqrt{e} \times 2R = (CF \&c.) = 4R^2e\sqrt{2}$ = Sup. of the <i>Rhom.</i> $FCBE$. 45
But	}	3	$4R^2e =$ Superficies of the <i>Sphere</i> $HGDI$.
2, 3	}	4	$4R^2e : 4R^2e\sqrt{2} ::$ Superf. <i>Sphere</i> : Sup. <i>Rhom.</i> :: $2R : 2R\sqrt{2} :: FC : FB$.

Part

Part II.

For	{	1	$2 R \sqrt{2} = BF = CE.$
		2	$2 R^2 e = \text{Sup. upon Circle } CE, \text{ the common Base.}$
		3	$\frac{4 R^3 e}{3} \sqrt{2} = \text{Solidity of the Rhomb. } FCBE.$
		4	$\frac{4 R^3 e}{3} = \text{Solidity of the Sphere } HGDI, \text{ 101.}$
But		5	$\frac{4 R^3 e}{3} : \frac{4 R^3 e \sqrt{2}}{3} :: 2 R : 2 R \sqrt{2} :: FC : FB.$

COROLLARY.

410. Hence it is evident, that the Solidities of Fig. 29. the *Sphere* and *Rhombus* circumscribed about it, are in Proportion one to another, as their Superficies.

For	{	1	$\frac{4 R^3 e}{3} : \frac{4 R^3 e \sqrt{2}}{3} :: FC : FB. \quad 409.$
		2	$4 R^2 e : 4 R^2 e \sqrt{2} :: FC : FB. \quad 409.$
		3	$\frac{4 R^3 e}{3} : \frac{4 R^3 e \sqrt{2}}{3} :: 4 R^2 e : 4 R^2 e \sqrt{2}$ per 16.6, or 11.5

1 ÷ 3

But

PROP.

P R O P. XXXV.

Fig. 30. 411. If there be two *Isoceles* Cones ABC , XZ , and the Superficies of the Cone ABC , be equal to Z the Base of the other; and the Altitude X , equal to a Perpendicular DE , drawn to AC the side of the Cone ABC , these Cones will be equal.

For	1	$\frac{BC^2 \times e}{4} = \text{Superf. Circle } BC = \text{Base of the Cone } BAC. \quad 31.$
	2	$BC \times e \times \frac{AC}{2} = \text{Superf of the Cone } BAC = Z. \quad 45.$
But	3	$AD : AC :: AE : AD, \text{ per } 4.6.$
3 ∴	4	$AE = \frac{AD^2}{AC}$
	5	$\frac{AD^2}{AC} (= AE) : DE :: AD : \frac{BC}{2} (= DC) \text{ per } 4.6.$
5 ∴	6	$DE = \frac{AD \times BC}{2 AC} = X, \text{ per Hypoth.}$
But 1,2,6	7	$BC^2 \times e : BC \times e \times \frac{AC}{2} :: \frac{AD \times BC}{2 AC} : AD, \text{ per } 16.6.$ Therefore Cone $BAC = \text{Cone } XZ$ per <i>Art.</i> 84 and 96.

P R O P

P R O P. XXXVI.

412. Every *Rhombus GAHD*, composed of *Isoceles Cones GAH, DGH*, is equal to a *Cone XZ*, whose *Base Z*, is equal to the *Superficies* of the *Cone GAH*, and its *Altitude X*, equal to *DE*, a *Perpendicular* drawn from the *Vertex D*, of the *Cone DGH*, to the *side AH*, of the first *Cone*.

For	{	1	$\frac{\overline{HG}^2 \times e}{4}$ = Superficies of the Circle GH the common Base. 31.
		2	$HG \times e \times \frac{AH}{2}$ = Sup. Cone GAH, (per Art. 45.) = Z, per Hyp.
But	{	3	$AH : \frac{HG}{2} (= HK) :: AD : DE$, per 4.6.
3 ∴		4	$\frac{HG \times AD}{2 AH} = DE = X$, per Hyp.
But 1.2.4.	{	5	$\frac{\overline{HG}^2 \times e}{4} : HG \times e \frac{AH}{2} :: \frac{HG \times AD}{2 AH}$: AD, per 16.6
			Therefore the <i>Rhombus GAHD</i> = <i>Cone XZ</i> , per Art. 84. 96.

P R O P. XXXVII.

Fig. 31.

415. If an Isosceles Cone BAC , be cut by a plain GH , parallel to the Base BC ; and if you describe the Cone GH , whose Base is the plain BH , and Vertex D the Center of the Base; and if the *Rhombus* $AGDH$, be taken from the Cone BAC , the Remainder will be equal to a Cone ZX , having its Base Z , equal to the conical Superficies $GBCH$, intercepted by parallel Lines and its Altitude X , equal to DE , a Perpendicular drawn from the Center D to the side of the Cone.

For	{	1	$BC \times e \times \frac{AC}{2} \times \frac{DE}{3} = \text{Solidity of the Cone } BAC. \quad 411.$
		2	$HG \times e \times \frac{AH}{2} \times \frac{DE}{3} = \text{Solidity of the Rhombus } AGDH. \quad 412.$
1-2		3	$BC \times e \times \frac{AC}{2} - HG \times e \times \frac{AH}{2} \times \frac{DE}{3} = \text{Cone } BAC - \text{Rhombus } GHAD.$
But		4	$BC \times e \times \frac{AC}{2} - GH \times e \times \frac{AH}{2} = Z, \text{ per (Hyp.)}$
$4 \times \frac{DE}{3} = X$		5	$BC \times e \times AC - HG \times e \times \frac{AH}{2} \times \frac{DE}{3} = \text{Cone } ZX.$
$3 = 5$		6	$\text{Cone } BAC - \text{Rhomb } AGDH = \text{Cone } ZX. \quad 54.$

414. If

P R O P. XXXVIII.

414. If one Cone AGH , of a *Rhombus* $AGDH$, Fig. 35, composed of *Isoceles* Cones GAH , $G D H$, be cut by a plain MN , parallel to the Base GH , and upon the Circle MN , the Cone MDN be described; having its Vertex D , the same with the Cone GHD ; and if from the whole *Rhombus* $GAHD$, the *Rhombus* $MNAD$ be taken away; the Remainder is equal to a Cone, whose Base Z , is equal to the conical Superficies $MGHN$, intercepted between the parallel Plains MN , GH , and its Altitude X , equal to DE , a Perpendicular drawn from the Vertex D to the side AH of the Cone GAH .

For	{	1	Superf. Cone $GAH \times \frac{DE}{3} = \text{Solid.}$ <i>Rhombus</i> $GAHD$. 412.
		2	Superf. of the Cone $MAN \times \frac{DE}{3}$ $= \text{Sol. Rhombus } MNAD$. 412
		3	$\overline{GAH - MAN} \times \frac{DE}{3} = GAHD - MNAD$.
1—2			
But		4	$GAH - MAN = \text{Superf. } MGHN = Z$, per Hyp.
$4 \times \frac{DE}{3} = X$		5	$\overline{GAH - MAN} \times \frac{DE}{3} = \text{Solid. of the}$ Cone ZX .
3=5		6	Sol. <i>Rhomb.</i> $GAHD - \text{Sol. Rhomb.}$ $MNAD = \text{Solid. Cone } ZX$. 54.

P R O P. XXXIX.

Fig. 36. 415. Every Solid contained under a conical Superficies inscribed in a Sphere, as $A F H D C B G E$, is equal to a Cone K , having for its Base a Circle, whose Area is equal to all the conical Superficies of the Solid, and its Height equal to a Perpendicular ZX , drawn from Z the Center of the Sphere, to one side of the Polygon.

For $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right.$	1 Sol. of the Rhomb. $EXFA = \text{Sup. of}$ the Cone $EAF \times \frac{ZX}{3} \cdot 412$
	2 Sol. Rhomb. $G H X \mathcal{Q}$ — Rhomb. $EXF \mathcal{Q} = \text{Sup. of the Cone } EGHE$ $\times \frac{DE}{3} \cdot 414.$
	3 Sol. Rhomb. BPD — Rhomb. $G X H P$ $= \text{Sup. Cone } BGHD \times \frac{XZ}{3} \cdot 414$
	4 $\frac{2 \times XEFA + G X H \mathcal{Q} - E X F \mathcal{Q} +$ $B P D - G X H P (= \text{Sol. of the Poly-}$ $\text{gon per Fig.}) = 2 \times EAF + EHGF$ $+ BGHD (= \text{Sup. of Polygon}) \times$ $\frac{ZX}{3} = \text{Sol. of the Cone } K.$

$$\frac{1+2+}{3 \times 2}$$

The

The same another way demonstrated as follow-Fig. 21
eth, viz. put $2R = AE$, the Diameter of the
circumscribing Sphere, and $s = AB$, the side of
the Polygon.

$1 \times 2 R e$	1	$\sqrt{4R^2 - s^2} = BE$, per 47.1.
	2	$2 R e \sqrt{4R^2 - s^2} = AE \times BE \times e = I^2 e =$ Sup. of the Polygon K. 372.
	3	$2R : R :: \sqrt{4R^2 - s^2} : \frac{\sqrt{4R^2 - s^2}}{2}$ $= ZX$, per 4.6,
$2 \times \frac{1}{3}$ of the 3	4	$\frac{R e}{3} \times \sqrt{16R^4 - 8R^2 s^2 + s^4} = \text{Sol.}$ of the Cone $K \times \frac{ZX}{3}$; now sup. $s = 0$.
4 ..	5	$\frac{R e}{3} \sqrt{16R^4} = \frac{4R^3 e}{3} = \text{Solidity of}$ the Sphere.

And therefore the fourth Step must be the Ex-
pression for the Solidity of the Body.

P R O P. XL.

416. Every thing being supposed as in *Art.* 371,
I say that a Circle, the square of whose Radius is
equal to the Rectangle under one side AB or BC ,
and of all the joining Lines BH , CH , DF , taken
together, is equal to all the Conic Superficies in-
scribed in the Sphere.

For the Radius of a Circle equal to all the Conic Superficies put I .

$$\text{Then } \left\{ \begin{array}{l} 1 \mid AE \times BE = I^2, \text{ per 372.} \\ 2 \mid AE \times BE = \overline{BH + DF + CG} \times AB \\ 3 \mid \overline{BH + CG + DF} \times AB = I^2 \end{array} \right. \quad 369$$

P R O P. XLI.

Fig. 37. 417. If there be a polygonous Figure $EFGHIK$, &c. contained under conical Superficies circumscribed about a Sphere, its Superficies will be equal to a Circle, whose Radius is equal to a Rectangle contained under one side, EF , of the Polygon, and of all the joining Lines FM , GL , HK , taken together.

Describe a Circle upon the Center, N , of the Solid, which is also the Center of the Sphere; and let it pass through the Angles E, F, G, H, I, K, M , and the Proposition will be evident from *Art.* 369, and 372.

P R O P. XLII.

Fig. 37. 418. If there be a Figure, $ANBCD$, inscribed in a Sphere, and another, $EFGIL$, circumscribed about it, and both contained under an equal Number of like conical Superficies; the Superficies of the circumscribing Figure, is to the Superficies of the inscribed Figure in a duplicate Proportion of their sides FG , and BN , and the Solidity of the circumscribing Figure, is to the Solidity of the inscribed,

bed, in a triplicate Proportion of the same sides.

The first part is evident from *Art.* 106, and the latter part from *Art.* 105.

P R O P. XLIII.

419. The Superficies of the Portion *B G H D*, Fig. 38. which contains an equilateral Cone, *B K D*, is double to the Superficies of the said Cone.

For	{	1, 2		1	Circle on $2BK = \text{Sup. Segm. Sphere } BDK.$ 383.
				2	Circle on $2BK : \text{Sup. Cone } BDK :: BK : AB.$ 364.
				3	Sph. Segm. <i>BDK</i> : Sup. Cone <i>BDK</i> :: $BK : AB :: 2 : 1.$

P R O P. XLIV.

420. The Superficies of a Sphere, is to the whole Superficies of an equilateral Cone inscribed in it, as 16 to 9.

Put $KO = 2R$, then is $BD = BK = R\sqrt{3}$, *Cor.* 1. 8. 6. &c.

Then	{	1		$\frac{Re\sqrt{3}}{2} \times R\sqrt{3} + \frac{3R^2e}{2} = \frac{9R^2e}{4} =$
				whole Sup. Cone. 45.
But	{	2		$4R^2e = \text{Superf. of the Sphere}$ 103
				3

P R O P.

P R O P. XLV.

421. The Superficies of a Sphere, bears that Proportion to the whole Superficies of an equilateral Cone circumscribed about it, that 4 bears to 9.

Fig. 24.

N. B. Here $NO = KB$, as is plain per Figure; therefore if for NO we put $4R$, we shall have $KB = 2R$.

$$\text{For } \left\{ \begin{array}{l|l} 1 & 4 R^2 e = \text{Sup. of the Sph. } KMBA. \quad 103 \\ 2 & 9 R^2 e = \text{whole Sup. of Cone } DOF. \quad 45 \\ 1, 2 & 3 & 4 R^2 e : 9 R^2 e :: 4 : 9, \text{ per 16.6} \end{array} \right.$$

COROLLARY.

422. From this Demonstration it is evident, that the Axis, BO , of the equilateral Cone, circumscribed about a Sphere, is to the Diameter of the Sphere BK , as 3 to 2.

P R O P. XLVI.

Fig. 23. 24. 423. The Base QI of the Cone DOF , which circumscribes a Sphere $BPKM$, is to the Base ZT of the circumscribing Cylinder, as 3 to 1.

The same things being supposed as in the last.

$$\text{For } \left\{ \begin{array}{l|l} 1 & 3 R^2 e = \text{Circle } QI, (\text{per Art. 31}) \\ & = \text{Sup. of Cone's Base } DOF. \quad 31 \\ 2 & R^2 e = \text{Circle } ZT = \text{Basis of the} \\ & \text{Cylinder } PHVS. \\ \text{But } 1, 2 & 3 & 3 R^2 e : R^2 e :: 3 : 1, \text{ per 16.6.} \end{array} \right.$$

P R O P.

P R O P. XLVII.

424. The Superficies of an equilateral *Cone* *DOF* Fig. 24, circumscribed about a *Sphere*, is to the Superficies of an equilateral *Cylinder* circumscribed about the same, as 3 to 2.

Every thing remaining as before.

For	1	$R\sqrt{3} = BD$, per 13.6.
$1 \times 2e$	2	$2Re\sqrt{3}$ = circumference of the Circle <i>QI</i> . 30.
$2 \times R\sqrt{3} = \frac{DO}{\bullet}$	3	$6R^2e$ = Superficies of the <i>Cone</i> <i>DOF</i> . 45.
And	4	$4R^2e$ = Sup: of the <i>Cylinder</i> 42
But	5	$6R^2e : 4R^2e :: 3 : 2$, per 16.6

P R O P. XLVIII.

425. The whole Superficies of an equilateral *Cone* circumscribed about a *Sphere*, is quadruple to the whole Superficies of an equilateral *Cone* inscribed in the same *Sphere*.

For	1	Sup. $\{DOF : \}$ Sup. $\{ : : 9 : 4. \}$ 421
	2	Cone $\{SKT : \}$ <i>Sphere</i> $\{ : : 9 : 16. \}$ 420
1, 2	3	Sup. <i>Cone</i> <i>DOF</i> : Sup. <i>Cone</i> <i>SKI</i> : : 16 : 4 : : 4 : 1. per 11.5.

D d

P R O P.

P R O P. XLIX.

Fig. 24. 426. A Sphere hath that Proportion to *DOF* an equilateral *Cone* inscribed in it, which 32 hath to 9.

For the Diameter *NO*, put $2R$.

For	{	1	$\frac{4R^3e}{3} = \text{Solid. of the Sphere.}$	101
		2	$\frac{3R^2e}{4} = \text{Sup. of the Circle } QT.$	31
$2 \times \frac{R}{2}$		3	$\frac{3R^3e}{8} = \text{Solid. of the Cone } DOF,$	97
1, 3		4	$\frac{4R^3e}{3} : \frac{3R^3e}{8} :: 32 : 9, \text{ p. 16.6.}$	

P R O P. L.

Fig. 24. 427. An equilateral *Cone* circumscribed about a *Sphere*, is eight-fold of an equilateral *Cone* inscribed in the same *Sphere*.

For	{	1	<i>Sph. DOFN</i> : <i>Cone DOF</i> :: 32 : 9, 426
		2	<i>Sph. SKMB</i> : <i>Cone SKI</i> :: 32 : 9, 426
Then		3	<i>Sph. DOFN</i> : <i>Sph. SKMB</i> :: <i>Cone DOF</i> : <i>SKI</i> , per 11.5.
Suppose		4	$ON = 2R$, then will $BK = R = \text{Diameter Sphere } SKMB.$
Then		5	<i>Sph. DOFN</i> : <i>Sph. SKMB</i> :: $8R^3$: R^3 :: 8 : 1, 104.
3, 5		6	<i>Cone DOF</i> : <i>Cone SKI</i> :: 8 : 1, per 11.5

The

The last shorter, thus ;

The	$\left\{ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right\}$	$\left\{ \begin{array}{c} OB \\ KC \end{array} \right\}$	of the Cone	$\left\{ \begin{array}{c} DOF = \frac{2}{3} \frac{R}{B} \\ SKI = \frac{3}{8} \frac{R}{B} \end{array} \right\}$	108.
Altit.					
But					
					$Cone DOF : Cone SKI :: \frac{8 R^3}{27} : \frac{R^3}{27}$
					$:: 8 : 1, \quad 104.$

P R O P. LI.

428. A *Sphere* hath the same Proportion, both in Fig. 24. Respect of Solidity and Superficies, to the equilateral *Cone D O F* circumscribed about it, which 4 hath to 9.

For	$\left\{ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right\}$	<i>Cone DOF</i> : <i>Cone SKI</i> :: 8 : 1 :: 32 : 4 427
Theref.		<i>Sph. SKMB</i> : <i>Cone SKI</i> :: 32 : 9. 426
		<i>Sph. SKMB</i> : <i>Cone DOF</i> :: 4 : 9, p. 115

The second Part is plain from *Art.* 421.

SCHOLIUM.

429. That therefore which *Archimedes* was surpris'd at, in a *Sphere* and *Cylinder* encompassing it, we have also now demonstrated in a *Sphere* and an equilateral *Cone* encompassing it ; to wit, that there is the same rational Proportion of the Solidities between themselves, which there is of the Surfaces : For as he found that the *Sphere* is to the *Cylinder*,

D d 2

as

P R O P. LIII.

431. The same sesquialteral Proportion holds between an Equilateral *Cone* and *Cylinder*, circumscribed about the same *Sphere*, in respect of their whole Surfaces, their simple Surfaces, their Solidities, Altitudes, and Bases,

This *Proposition* is manifest, as to the whole Surfaces and Solidities, from the foregoing; and as to the simple Superficies from *Art.* 424; and as to their Altitudes and Bases from *Art.* 422, and 423.

P R O P. LIV.

432. If in a Circle $BGRQ$, there be drawn the Fig. 23.
Chords QR , QB , QG , the Superficies of the *Cone*, made by the Revolution of the Triangle GQB turned round about the Chord G , will be a mean Proportional between the Superficies of the spherical Segments QBG , QRB , made by the Revolution of the Circle round the Diameter BG .

For

For $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \left\{ \begin{array}{l} \overline{GQ}^2 \times e \\ \overline{BQ}^2 \times e \end{array} \right\} = \text{Supplement of the Seg-}$
 $\text{ment } \left\{ \begin{array}{l} 2RG \\ 2RB \end{array} \right\} 383$
 $3 \quad 2 \quad 2B \times e \times \frac{1}{2} GQ (= 2B \times e \times$
 $GQ,) = \text{Superficies of the}$
 $\text{Cone } GQB. 45$

But $\left\{ \begin{array}{l} 4 \\ \end{array} \right\} \left\{ \begin{array}{l} \overline{GQ}^2 \times e : BQ \times e \times GQ :: BQ \\ \times e \times GQ : \overline{BQ}^2 \times e, \text{ per} \end{array} \right\}$

(16.6)



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APPENDIX;

Containing several curious

PROBLEMS,

Concerning the

TRANSMUTATION

O F

SOLIDS:

Or finding the Diameters, Sides,
&c. of one *SOLID*, whose *SU-*
PERFICIES, *SOLIDITY*, &c. is
equal, or bears a given Proportion to the
SUPERFICIES, *SOLIDITY*, &c.
of another *SOLID* given.

Printed in the Year 1732.



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APPENDIX.

Concerning the

TRANSMUTATION

O F

S O L I D S, &c.

P R O B L E M I.

TO find a Circle, whose Superficies is equal to the Superficies of a *Cylinder* given.

For the Length of the *Cylinder* put A , for the Diameter of its Base $2R$, and for the Diameter of the Circle sought put $2r$.

$$\begin{array}{l} \text{Then } \left\{ \begin{array}{l} 1 \mid r^2 e \\ 2 \mid 2R e A \end{array} \right\} = \text{Sup. } \left\{ \begin{array}{l} \text{Circle} \\ \text{of the } \text{Cylinder} \end{array} \right. \begin{array}{l} 31 \\ 42 \end{array} \\ 1 = 2 \\ 3 \text{ red. } u2 \mid \begin{array}{l} 3 \mid r^2 e = 2 R e A, \text{ per Hypothef.} \\ 4 \mid r = \sqrt{2 R A}. \end{array} \end{array}$$

F. e

T H E C.

T H E O R E M.

Find a mean Proportional between the Diameter of a *Cylinder* and its Length, and that will be the Radius of the Circle sought.

If a Circle whose Superficies is equal to the whole Superficies of the *Cylinder* be required, every thing continuing as before, we shall

$$\begin{array}{lcl}
 \text{have} & \left\{ \begin{array}{l} 1 \left| r^2 e \right. \\ 2 \left| 2 R e \times \overline{A+R} \right. \end{array} \right\} = \text{Sup.} & \left. \begin{array}{l} \text{Circle} \\ \text{of the} \\ \text{Cylinder} \end{array} \right\} \begin{array}{l} 31 \\ 43 \end{array} \\
 1 = 2 & \left\{ \begin{array}{l} 3 \left| r^2 e = 2 R e \times \overline{A+R}, \text{ per Hyp.} \right. \\ 4 \left| r = \sqrt{2 R \times \overline{A+R}} \right. \end{array} \right. & \\
 3 \text{ Red, \&c.} & &
 \end{array}$$

T H E O R E M.

Find a mean Proportional between the Diameter of the *Cylinder*, and the Sum of the Radius, and Length thereof, and that will be the Radius of the Circle sought.

P R O B. II.

To find a *Cylinder*, whose Superficies is equal to the Superficies of a Circle given.

In this *Problem* are two Cases ; for first, the Length of the *Cylinder* may be given to find the Diameter ; or secondly, the Diameter may be given to find the Length.

C A S E

C A S E I.

The Length given to find the Diameter.

$$\begin{array}{l} \left| \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} 2 A R e = r^2 e, \text{ per Hyp.} \\ 2 R = \frac{r}{A}. \end{array} \\ 1 \div A e \end{array}$$

T H E O R E M.

Divide the Square of the Radius of the Circle by the Length of the *Cylinder*, and the Quotient will be the Diameter.

C A S E II.

The Diameter of the Cylinder given, to find the Length.

$$\begin{array}{l} \left| \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} 2 R A e = r^2 e, \text{ per Hyp.} \\ A = \frac{r^2}{2 R}. \end{array} \\ 1 \div 2 R e \end{array}$$

T H E O R E M.

Divide the Square of the Radius of the Circle, by the Diameter of the *Cylinder*, and the Quotient is the Length of the *Cylinder*.

Again ; if it be required to find a *Cylinder* whose whole Superficies is equal to the Superficies of a Circle given.

CASE I.

The Length of the Cylinder given, to find the Diameter.

Then	1	$2R^2e + 2ARe = r^2e$, per Hyp.
$1 \div 2e$	2	$R^2 + AR = \frac{r^2}{2}$.
$2e \square 2$	3	$R^2 + AR + \frac{1}{4}A^2 = \frac{A^2 + 2r^2}{4}$,
$3 \text{ w } 2$	4	$R + \frac{1}{2}A = \sqrt{\frac{A^2 + 2r^2}{2}}$.
$4 - \frac{A}{2}$	5	$R = \frac{\sqrt{A^2 + 2r^2} - A}{2}$.
5×2	6	$2R = \sqrt{A^2 + 2r^2} - A$.

THEOREM.

From the Square of the *Cylinder's* Length, and twice the Square of the Radius of the Circle proposed added together, extract the Square Root; from which subtract the Length of the *Cylinder*, and the Remainder will be the Diameter of the *Cylinder*.

CASE

C A S E II.

The Diameter of the Cylinder given, to find the Length.

Then,	$\left \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right $	$2 A R e + 2 R^2 e = r^2 e$, per Hyp.
$1 \div 2e$		$A R + R^2 = \frac{r^2}{2}$.
$2 - R^2$		$A R = \frac{r^2}{2} - R^2$.
$3 \div R$		$A = \frac{r^2}{2R} - R$.

T H E O R E M.

Divide the Square of the Radius of the Circle proposed, by the Diameter of the Cylinder; from that Quotient subtract the Radius of the Cylinder, and the Remainder will be the Length of the Cylinder.

P R O B. III.

To find a Circle whose Superficies is equal to the Superficies of a Cone.

$$\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \left\{ \begin{array}{l} r^2 e - \\ R e S \end{array} \right\} = \text{Sup. of the } \left\{ \begin{array}{l} \text{Circle} \\ \text{Cone} \end{array} \right. \begin{array}{l} 31. \\ 45. \end{array}$$

$$r^2 e = R e S, \text{ i. e. } r = \sqrt{R S}.$$

T H E O.

T H E O R E M.

Find a mean Proportional between the Radius of the Cone's Base, and Side of the Cone, that is the Radius of the Circle sought.

If it were required to find a Circle equal to the whole Superficies of the Cone.

$$\begin{array}{l} \text{For} \\ 1=2 \\ 3 \div e \& w \end{array} \left\{ \begin{array}{l} 1 \left| r^2 e \right. \\ 2 \left| R e \times \overline{s+R} \right. \\ 3 \left| r^2 e = R e \times \overline{s+R}, \text{ per Hyp.} \\ 4 \left| r = \sqrt{R \times \overline{s+R}}. \right. \end{array} \right\} = \text{Superf.} \left\{ \begin{array}{l} \text{Circle 31} \\ \text{of the Cone 45} \end{array} \right.$$

T H E O R E M.

Find a mean Proportional between the Radius of the Cone, and the Sum of the Radius and Length of the Cone, and that will be the Radius of the Circle.

P R O B. IV.

To find a Cone whose whole Superficies is equal to a Circle given.

In this *Problem* are two *Cases*; for first, the Side of the Cone may be given to find the Diameter of its Base; or secondly, the Diameter may be given to find the Side.

C A S E

C A S E I.

The Side given to find the Diameter.

$1 \div e$	1	$R e \times \overline{S + R} = r^2 e, \text{ p. Hyp. 45.31}$
	2	$R^2 + SR = r^2$
$16 \square$	3	$R^2 + SR + \frac{S^2}{4} = r^2 + \frac{S^2}{4}$
$3 \text{ w } 2$	4	$R + \frac{S}{2} = \sqrt{\frac{4r^2 + S^2}{2}}$
$4 - \frac{S}{2} \times \times 2$	5	$2 R = \sqrt{4r^2 + S^2} \div S.$

T H E O R E M.

To the Square of the Circle's Diameter, add the Square of the Side of the Cone ; from the Square-Root of that Sum subtract the Side of the Cone, and the Remainder will be the Diameter of the Cone.

C A S E II.

The Diameter of the Cone given, to find the Side.

$1 - R^2$	1	$R^2 + SR = r^2, \text{ per Hypoth.}$
	2	$SR = r^2 - R^2,$
$2 \div R$	3	$S = \frac{r^2}{R} - R.$

T H E O-

T H E O R E M.

From the Square of the Circle's Radius, divided by the Radius of the Cone, subtract the said Radius; the Remainder is the side.

P R O B. V.

To find the Diameter of a Sphere, whose Superficies is equal to the Superficies of a given Cylinder.

For the Diameter of the Sphere put $2r$, and all the rest as before.

$$\begin{array}{l} \text{Then} \left\{ \begin{array}{l} 1 \left| 4r^2 e \right. \\ 2 \left| 2ARe \right. \end{array} \right\} = \text{Superf. of the } \left\{ \begin{array}{l} \text{Sphere } 103 \\ \text{Cylinder } 42 \end{array} \right. \\ 1 = 2 \quad \left| \begin{array}{l} 3 \left| 4r^2 e = 2ARe, \text{ per Hyp.} \\ 4 \left| 4r^2 = 2AR. \\ 5 \left| 2r = \sqrt{2RA}. \end{array} \right. \right. \\ 3 \div e \\ 4 \text{ as } 2 \end{array}$$

T H E O R E M.

Find a mean Proportional between the Diameter of a Cylinder and its Length, and it will be the Diameter of the Sphere.

If a Sphere whose Superficies is equal to the whole Superficies of the Cylinder be required:

$$\begin{array}{l} \text{Then} \left| \begin{array}{l} 1 \left| 4r^2 e = 2RE \times \overline{A+R}, \text{ per Hyp.} \\ 2 \left| 4r^2 = 2R \times \overline{A+R}. \\ 3 \left| 2r = \sqrt{2R \times \overline{A+R}}. \end{array} \right. \right. \\ 1 \div e \\ 2 \text{ as } 2 \end{array}$$

T H E O-

T H E O R E M.

A mean Proportional between the Diameter of a Cylinder, and the Sum of its Radius and Length, will be the Diameter of a Sphere, &c.

P R O B. VI.

To find a Cylinder, whose whole Superficies is equal to the Superficies of a given Sphere.

In this *Problem* are two *Cases*.

C A S E I.

The Length of the Cylinder given, to find the Diameter.

$$\begin{array}{l|l}
 1 \div 2e & 1 \left| \begin{array}{l} 2R^2e + 2ARe = 4r^2e, \text{ per Hyp.} \\ 2 \left| \begin{array}{l} R^2 + AR = 2r^2, \\ 3 \left| \begin{array}{l} R^2 + AR + \frac{1}{4}A^2 = 2r^2 + \frac{1}{4}A^2, \\ 4 \left| \begin{array}{l} R + \frac{1}{2}A = \sqrt{\frac{8r^2 + A^2}{2}}, \\ 5 \left| \begin{array}{l} 2R = \sqrt{8r^2 + A^2} \div A. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \\
 2 \text{ } \square & \\
 3 \text{ } \omega & \\
 4 - \frac{1}{2}A, \&c &
 \end{array}$$

T H E O R E M.

To twice the Square of the Diameter of the given Circle, add the Square of the Length of the Cylinder; from the Square Root of that Sum, subtract the said Length, and the Remainder will be the Diameter of the Cylinder.

F f

C A S E

C A S E II.

The Diameter given, to find the Length.

$$\begin{array}{l|l|l} 1 - R^2 & 1 & R^2 + AR = 2r^2, \text{ per Hyp.} \\ & 2 & AR = r^2 - R^2. \\ 2 \div R & 3 & A = \frac{r^2}{R} - R. \end{array}$$

T H E O R E M.

Divide the Square of the Radius of a Circle, by the Radius of the Cylinder, and from the Quotient subtract the said Radius, and the Remainder will be the Length of a Cylinder equal to the Circle.

P R O B. VII.

To find a square Parallelepipedon, whose Length, and whole Superficies, is equal to the Length and whole Superficies of a given Cylinder.

$$\begin{array}{l|l|l} \text{For } \{ & 1 & 2L \times \overline{L+2A} \\ & 2 & 2Re \times \overline{R+A} \} = \text{whole Superf. of the} \\ & & \begin{array}{l} \text{Parallelepipedon} \quad 35. \\ \text{Cylinder.} \quad 42. \end{array} \\ 1=2 & 3 & 2L \times \overline{L+2A} = 2Re \times \overline{R+A}, \text{ per Hyp.} \\ 3 \div 2, \&c & 4 & L^2 + 2AL = Re \times \overline{R+A}. \\ 4 \square & 5 & L^2 + 2AL + A^2 = A^2 + Re \times \overline{R+A}. \\ 5 \text{ w } 2 & 6 & L + A = \sqrt{A^2 + Re \times \overline{R+A}}. \\ 6 - A & 7 & L = \sqrt{A^2 + Re \times \overline{R+A}} \div A. \end{array}$$

T H E O

T H E O R E M.

To the Square of the common Length, add half the common Superficies, and from the Square Root of that Sum, subtract the common Length, and the Remainder will be the Side of the Parallelepipedon sought.

P R O B. VIII.

To find the Length of a Square Parallelepipedon, whose Side is equal to the Diameter of a Cylinder given; and its Superficies equal to the Superficies of the said Cylinder.

For the Length of the Parallelepipedon put S .

$$\begin{array}{l|l} \text{Then } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right. & \begin{array}{l} 4LS = 8RS, \text{ (because } L = 2R \text{)} \\ \quad = \text{Superf. Parallelep.} \quad 36. \\ 2ARe = \text{Super. of the Cylinder } 42. \\ 4RS = ARe. \\ S = \frac{Ae}{4} \end{array} \end{array}$$

$1 = 2$

$3 \div 4R$

T H E O R E M.

Take a fourth Part of the Rectangle of the Length of the Cylinder, and Ratio of the Circumference of a Circle to its Diameter, for the Length of the Parallelepipedon.

E f 2

P R O B.

P R O B. IX.

To find a Parallelepipedon, whose Length and Superficies, are equal to the Length and Superficies of a Cylinder given; and its Sides in the Ratio of a to b .

	1	$a : b :: L : B$, per given Ratio.
1 $\therefore$	2	$\frac{Lb}{a} = B$.
Then,	3	$L + \frac{Lb}{a} \times 2A$
	4	$2ARe$
		$\left. \begin{array}{l} \text{Parallelepipedon} \\ \text{Cylinder.} \end{array} \right\} = \text{Superficies of}$
		$\left. \begin{array}{l} \text{the} \\ \text{Cylinder.} \end{array} \right\} \begin{array}{l} 35 \\ 42. \end{array}$
	5	$L + \frac{Lb}{a} \times 2A = 2ARe$, per
		(Hyp.
$5 \div 8 \times$	6	$La + Lb = Rae$.
$6 \div a + b$	7	$L = \frac{Rae}{a + b}$.

T H E O R E M.

Multiply half the Circumference of the Cylinder, into one of the Terms of the Ratio of the Sides, and divide that Product by the Sum of the said Terms,

Terms, and the Quotient will be one of the Sides of the Parallelepipedon. So having found one of the Sides by this *Theorem*, the other Side may be found by the Ratio given.

N. B. If you multiply the half Circumference by the greater Term of the Ratio, this *Theorem* will give the greater Side; and if you multiply by the lesser Term, it will give the lesser Side.

Suppose it were required to find a Parallelepipedon, whose Length is equal to the Length of a given Cylinder, and its Sides in the given Ratio, and its whole Superficies equal to the whole Superficies of the said Cylinder.

	1	$2L + \frac{2Lb}{a} \times A + \frac{2L^2b}{a} = 2Re$
		$\times \overline{A+R}$, per Hyp.
$1 \div 2 \times a$	2.	$La + Lb \times A + L^2b = Rea \times \overline{A+R}$; make $A \times \frac{a+b}{b} = rx$
	3	$L^2 + 2xL = Rea \times \overline{A+R}$,
$3 \text{ } \square$	4	$L^2 + \frac{2xL}{\overline{A+R}} x^2 = x^2 + Rea \times \overline{A+R}$
$4 \text{ } \omega 2$	5	$L + x = \sqrt{x^2 + Rea \times \overline{A+R}}$,
$5 - x$	6	$L = \sqrt{x^2 + Rea \times \overline{A+R}} \div x$.

THEOREM.

Divide the Sum of the Terms of the Ratio by one of those Terms, and square half the Quotient, add that Square to the Rectangle of half the
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Elements of SOLID GEOMETRY

Superficies of the Cylinder multiplied into the contrary Term to that by which their Sum was divided; then half the foresaid Quotient taken from the Square Root of the last Sum, leaves the one of the Sides of the Parallelepipedon.

One of the Sides being found by this *Theorem*, the other may be found by the Ratio.

P R O B. X.

To find a *Tetraedron*, whose Superficies is equal to the whole Superficies of a given Cylinder.

$$\begin{array}{rcl}
 \begin{array}{l} 1 \\ 2 \end{array} & \left| \begin{array}{l} a^2 \sqrt{3} \\ 2 R e \times \overline{A + R} \end{array} \right. & \left. \right\} = \text{Superficies of} \\
 & & \text{the } \left\{ \begin{array}{l} \textit{Tetraedron} \\ \textit{Cylinder} \end{array} \right. \quad \begin{array}{l} 114. \\ 43. \end{array} \\
 1,2 & 3 & a^2 = \frac{2 R e \times \overline{A + R}}{\sqrt{3}} \\
 3 \text{ or } 2 & 4 & a = \sqrt{\frac{2 R e \times \overline{A + R}}{\sqrt{3}}}
 \end{array}$$

T H E O R E M.

Divide the whole Superficies of any *Cylinder* by the Square Root of 3, the Square Root of that Quotient will be the Side of a *Tetraedron*, whose Side, &c.

P R O B.

P R O B. XI.

Let it be required to find the Length of a *Cylinder*, whose whole Superficies is equal to the Superficies of a *Tetraedron* inscribed in a *Sphere*, whose Diameter is equal to the Diameter of the *Cylinder*.

$$\begin{array}{l|l}
 1 & 2 R e \times \overline{A + R} \\
 2 & \frac{8 R^2}{\sqrt{3}}
 \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} = \text{Superficies of} \\
 \text{the } \left. \vphantom{\begin{array}{l} \text{Cylinder} \\ \text{Tetraedron} \end{array}} \right\} \begin{array}{l} 43. \\ 115. \end{array}$$

$$\begin{array}{l|l}
 1 = 2 & 3 \quad 2 R e \times \overline{A + R} = \frac{8 R^2}{e \sqrt{3}}, \text{ per Hypoth.} \\
 3 \div 2 R & 4 \quad e \times \overline{A + R} = \frac{4 R}{e \sqrt{3}} \\
 4 \div e \times - R & 5 \quad A = \frac{4 R}{e \sqrt{3}} - R.
 \end{array}$$

T H E O R E M.

Divide 4 times the Diameter of the Sphere or Cylinder, by the Square Root of 3, multiplied into the Ratio of the Diameter of a Circle to its Circumference; from that Quotient subtract the Radius of the Sphere or Cylinder, and the Remainder will be the Length of the *Cylinder*.

P R O B.

P R O B. XII.

To find the Diameter of a *Sphere*, which contains a *Tetraedron*, whose Side is equal to the Side of a given *Octaedron*,

$$\begin{array}{l} \text{For} \\ 2 \text{ Redu.} \end{array} \left\{ \begin{array}{l} 1 \left| \frac{2 R \sqrt{2}}{\sqrt{3}} = \text{Side of the Tetraedron,} \right. \\ \quad \text{in the Term of the Sphere. } 109 \\ 2 \left| \frac{2 R \sqrt{2}}{\sqrt{3}} = Z, \text{ per Hypoth.} \\ 3 \left| 2 R = Z \sqrt{\frac{1}{2}}. \right. \end{array} \right.$$

T H E O R E M.

Multiply the Side of the *Octaedron* into the Square Root of three halves of an Unite, and the Product will be the Diameter of the *Sphere*.

P R O B. XIII.

To find the Diameter of a *Sphere*, which will contain an *Octaedron*, whose Superficies is equal to the Superficies of a given *Tetraedron*.

$$\begin{array}{l} 1 = 2 \\ 3 \text{ Red. \&c.} \end{array} \left\{ \begin{array}{l} 1 \left| 4 R^2 \sqrt{3} \right. \} = \text{Superf. } \left\{ \begin{array}{l} \text{Octaed. } 144 \\ \text{of the Tetraed. } 114 \end{array} \right. \\ 2 \left| a^2 \sqrt{3} \right. \\ 3 \left| 4 R^2 \sqrt{3} = a^2 \sqrt{3}, \text{ per Hyp.} \\ 4 \left| 2 R = a. \right. \end{array} \right.$$

T H E O.

T H E O R E M.

A Sphere whose Diameter is equal to the side of a *Tetraedron*, will contain an *Octaedron*, whose Superficies is equal to the Superficies of the *Tetraedron*.

P R O B. XIV.

To find the Diameter of a Sphere, which will contain a *Tetraedron*, whose Area is equal to the Area of an *Octaedron* contained in a Sphere whose Diameter is $2R$.

For the Diameter of the Sphere sought, put $2r$.

$$\begin{array}{lcl}
 \begin{array}{l} 1 \\ 2 \end{array} & \left| \begin{array}{l} 4 R^2 \sqrt{3} \\ 8 r^2 \sqrt{3} \\ 3 \end{array} \right. & \left. \begin{array}{l} \\ \\ \end{array} \right\} = \text{Area of the } \left\{ \begin{array}{l} \text{Octaed.} \\ \text{Tetraed.} \end{array} \right. \begin{array}{l} 114. \\ 114. \end{array} \\
 1 = 2 & 3 & \frac{8 r^2 \sqrt{3}}{3} = 4 R^2 \sqrt{3}, \text{ per Hypoth.} \\
 3 \text{ Red.} & 4 & 4 r^2 = 6 R^2. \\
 4 \text{ w } 2 & 5 & 2 r = 2 R \sqrt{1.5}.
 \end{array}$$

T H E O R E M.

Multiply the Diameter of the Sphere containing an *Octaedron*, by the Square Root of one and a half, and that Product will be the Diameter required.

P R O B. XV.

To find the Diameter of a Sphere, whose Superficies is a mean Proportional between its own Solidity, and the Solidity of a Cube inscribed in a Sphere whose Diameter is $2R$.

For the Diameter of the Sphere sought put $2r$.

Then	1	$4r^2e = \text{Sup.}$	} of the Sphere fought. {	$\left. \begin{array}{l} 103. \\ 101. \end{array} \right\}$
	2	$\frac{4r^3e}{3} = \text{Solid.}$		
	3	$\frac{8R^3}{3\sqrt{3}} = \text{Solid. of Cube in the Terms of the circumscr. Sphere}$		128.
But	4	$\frac{4r^3e}{3} : 4r^2e :: 4r^2e : \frac{8R^3}{3\sqrt{3}}$		per (Hypoth.
4 ∴	5	$\frac{32R^3r^3e}{9\sqrt{3}} = 16r^4e^2.$		
5 Red.	6	$\frac{4R^3}{9\sqrt{3}} = 2re.$		
6 ÷ e	7	$\frac{4R^3}{9e\sqrt{3}} = 2r.$		

T H E O.

T H E O R E M.

Divide the Solidity of the Cube, by six times the Ratio between the Diameter of a Circle and its Circumference, and the Quotient will be the Diameter required.

P R O B. XVI.

To find the side of a *Tetraedron*, whose Superficies is equal to the Superficies of a given *Dodecaedron*.

$$\begin{array}{lcl}
 \left. \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} a^2 \sqrt{3} \\ \frac{15y^2 s}{S} \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} = \text{Superf. of the} \left\{ \begin{array}{l} \text{Tetraed. 114.} \\ \text{Dodecaed. 166.} \end{array} \right. \\
 1 = 2 \quad \left. \begin{array}{l} 3 \\ 4 \end{array} \right| \begin{array}{l} a^2 \sqrt{3} = \frac{15y^2 s}{S}, \text{ per Hyp.} \\ a = \sqrt{\frac{15y^2 s}{S\sqrt{3}}} . \end{array}
 \end{array}$$

T H E O R E M.

Divide the Superficies of the *Dodecaedron*, by the Square Root of 3, and the Square Root of that Quotient will be the side of the *Tetraedron* sought.

P R O B. XVII.

To find the side of a *Tetraedron*, whose Solidity is equal to the Solidity of a *Dodecaedron* whose side is given.

$$\begin{array}{lcl}
 & \left. \begin{array}{l} 1 \quad \frac{a^3}{6\sqrt{2}} \\ 2 \quad \frac{5sy^3}{2S^2} \times \sqrt{\frac{4S^2}{b} - 1} \end{array} \right\} = \text{Solidity of} \\
 & \text{the } \left\{ \begin{array}{l} \textit{Tetraedron.} \\ \textit{Dodecaedron.} \end{array} \right. & \begin{array}{l} 117. \\ 168. \end{array} \\
 1 = 2 & 3 \quad \frac{a^3}{6\sqrt{2}} = \frac{5sy^3}{2S^2} \times \sqrt{\frac{4S^2}{b} - 1}, \text{ per Hyp.} & \\
 3 \times 6\sqrt{2} & 4 \quad a^3 = 6\sqrt{2} \times \frac{5sy^3}{2S^2} \times \sqrt{\frac{4S^2}{b} - 1}. & \\
 4 \text{ uv } 3 & 5 \quad a = \sqrt[3]{6\sqrt{2} \times \frac{5sy^3}{2S^2} \times \sqrt{\frac{4S^2}{b} - 1}}. &
 \end{array}$$

T H E O R E M.

Multiply the Solidity of the *Dodecaedron*, by six times the Square Root of 2, and the Cube Root of that Product will be the side of the *Tetraedron* required.

P R O B.

P R O B. XVIII.

To find the Diameter of a *Cylinder*, whose Length, is given, and its whole Superficies is equal to the Superficies of a given *Tetraedron*.

	1	$2 R e \times \overline{R + A} = a^2 \sqrt{3}, \text{ per Hypoth.}$
1 Red.	2	$R^2 + RA = \frac{a^2 \sqrt{3}}{2e}.$
2 6 □	3	$R^2 + RA + \frac{1}{4} A^2 = \frac{a^2 \sqrt{3}}{2e} + \frac{1}{4} A^2.$
3 w 2	4	$R + \frac{1}{2} A = \sqrt{\frac{a^2 \sqrt{3}}{2e} + \frac{1}{4} A^2}.$
4 $-\frac{1}{2} A$	5	$R = \sqrt{\frac{a^2 \sqrt{3}}{2e} + \frac{1}{4} A^2} \div \frac{1}{2} A.$
5 $\times \frac{1}{2}$	6	$2 R = \sqrt{\frac{2a^2 \sqrt{3}}{e} + A^2} \div A.$

T H E O R E M.

Divide the double Area of the *Tetraedron*, by the Ratio of a Circle's Diameter to its Circumference; to that Quotient add the Square of the Cylinder's Length; from the Square Root of that Sum take the said Length, and the Remainder will be the Diameter of the *Cylinder*.

Sup-

Suppose the Diameter given, to find the Length of the *Cylinder*.

$$\begin{array}{l|l}
 1 - R^2 & 1 \quad R^2 + AR = \frac{a^2 \sqrt{3}}{2e}, \text{ per Hyp.} \\
 2 \div R & 2 \quad AR = \frac{a^2 \sqrt{3}}{2e} - R^2. \\
 & 3 \quad A = \frac{a^2 \sqrt{3}}{2Re} - R.
 \end{array}$$

THEOREM.

Divide the Superficies of the *Tetraedron*, by the Circumference of the *Cylinder*, and from that Quotient subtract the Radius, and the Remainder will be the Length sought.

PROB. XIX.

To find the side of an *Octaedron*, whose Superficies is equal to the whole Superficies of a *Prism* given.

$$\begin{array}{l|l}
 1 = 2 & 1 \quad \left. \begin{array}{l} 2 Z^2 \sqrt{3} \\ NB \times \overline{A+P} \end{array} \right\} = \text{Superf. } \left\{ \begin{array}{l} \text{Octaed.} \quad 143. \\ \text{Prism} \quad 38. \end{array} \right. \\
 3 \div 2\sqrt{3} & 2 \quad \\
 & 3 \quad 2 Z^2 \sqrt{3} = NB \times \overline{A+P}. \\
 & 4 \quad Z^2 = \frac{NB \times \overline{A+P}}{2\sqrt{3}}. \\
 4 \text{ as } 2 & 5 \quad Z = \sqrt{\frac{NB \times \overline{A+P}}{2\sqrt{3}}}.
 \end{array}$$

THE O

T H E O R E M.

Divide the whole Superficies of the *Prism*, by twice the Square Root of 3; the Square Root of that Quotient is the side of the *Octaedron* required.

P R O B. XX.

To find a *Prism*, whose whole Superficies is equal to the Superficies of an *Octaedron* whose side is given.

This *Problem* is productive of many curious *Cases*, which do not only arise from the Dimensions of the *Prism*, according as they may be given, or sought; but also, according to any Proportion that those Dimensions may have one to another. A few of these *Cases* I shall instance in, and leave the rest for the Learner's Exercise.

C A S E I.

Given N, B, P, to find A.

$$\begin{array}{l|l}
 1 \div NB & \begin{array}{l} 1 \quad NB \times \overline{A+P} = 2 Z^2 \sqrt{3}, \text{ per Hyp.} \\ 2 \quad A + P = \frac{2 Z^2 \sqrt{3}}{NB}. \end{array} \\
 2 - P & 3 \quad A = \frac{2 Z^2 \sqrt{3}}{NB} - P.
 \end{array}$$

T H E O.

T H E O R E M.

Divide the Superficies of the *Octaedron*, by the Perimeter of the *Prism*, and from that Quotient subtract the Perpendicular from the Center of the Base to the middle of a Side, and the Remainder will be the Length required.

C A S E II.

Given N, B, A , required P .

$$\begin{array}{l|l} 1 \div NB & \begin{array}{l} 1 \quad NB \times \overline{A + P} = 2Z^2\sqrt{3}. \\ 2 \quad \overline{A + P} = \frac{2Z^2\sqrt{3}}{NB}. \\ 3 \quad P = \frac{2Z^2\sqrt{3}}{NB} - A. \end{array} \end{array}$$

T H E O R E M.

Divide the Superficies of the *Octaedron*, by the Perimeter of the *Prism*, and from that Quotient subtract the Length, and the Remainder will be the Perpendicular, &c.

C A S E III.

Given N, A, P , required B .

$$\begin{array}{l|l} 1 \div N \times \overline{A + P} & \begin{array}{l} 1 \quad NB \times \overline{A + P} = 2Z^2\sqrt{3}, \text{ per Hyp.} \\ 2 \quad B = \frac{2Z^2\sqrt{3}}{N \times \overline{A + P}}. \end{array} \end{array}$$

T H E O-

T H E O R E M.

Divide the Superficies of the *Octaedron*, by the Sum of the Length and Perpendicular multiplied into the Number of sides, and the Quotient will be the Length of a Side of the *Prism*.

C A S E IV.

Now suppose none of these four Terms were given, but only the Ratio, or Proportion which some one of them, as suppose *N*, bears to the other three, to find the *Prism*.

$$\begin{array}{lcl}
 \text{Suppose} & \left\{ \begin{array}{l} 1 \quad Nx = B \\ 2 \quad NZ = P \\ 3 \quad Ny = A \end{array} \right. & \text{Quere } N^3. \\
 \text{Then is} & 4 \quad N^3 \times \overline{xZ + xy} = 2Z^2\sqrt{3}. & \\
 4 \div \overline{xZ + xy} & 5 \quad N^3 = \frac{2Z^2\sqrt{3}}{\overline{xZ + xy}}. & \\
 5 \text{ m } 2 & 6 \quad N = \sqrt[3]{\frac{2Z^2\sqrt{3}}{\overline{xZ + xy}}} = \sqrt[3]{\frac{2Z^2\sqrt{3}}{x \times \overline{Z + y}}} &
 \end{array}$$

T H E O R E M.

Divide the Superficies of the *Octaedron*, by the Rectangle of the Ratio of *N* to *B*, viz. *x*, into the Sum of the other two Ratio's, and the Cube Root of that Quotient will give *N*, and the rest may be found by their Ratio's.

H h

C A S E

CASE V.

Suppose the Ratio's as underneath, and it were required to find A .

Suppose	{	$\begin{array}{l} 1 \quad Ax = N \\ 2 \quad AZ = B \\ 3 \quad Ay = P \end{array}$	} <i>Quere</i> A^3 .
Then is,		$4 \quad A^2 \times x Z \times \overline{A + Ay} = 2 Z^2 \sqrt{3},$ (per Hypoth.	
4 ∴		$5 \quad A^3 \times x Z \times \overline{y + 1} = 2 Z^2 \sqrt{3}.$	
$5 \div x Z \times y + 1$		$6 \quad A^3 = \frac{2 Z^2 \sqrt{3}}{x Z \times y + 1}.$	
6 ∴		$7 \quad A = \sqrt[3]{\frac{2 Z^2 \sqrt{3}}{x Z \times y + 1}}.$	

THEOREM.

Multiply the Ratio of A , to P made more by 1, into the Rectangle of the other two Ratio's, and by that Product divide the Superficies of the *Octaedron*; then the Cube Root of that Quotient will be A , the Length of the *Prism*.

SCHOLIUM.

I might proceed to more *Cases*, almost *ad infinitum*; but the Excellency of the *Problem*, and its easy and obvious Application to any regular Body whatever, having already caused me to transgress due Bounds,

I shall only observe, that the Variety of *Cases* does not only consist in the different *Data*, and *Quæsitæ*, in the *Prism*, nor that the same *Cases* may with a little Variation be carried to the *Cylinder*, *Cone*, *Pyramid*, &c. which they really may be; but also that their Area's may be made equal, not only to the Area of any Solid Body whatever, but also to any given Number whatever, and the Operation will proceed just after the same manner.

P R O B. XXI.

To find the side of a *Dodecaedron*, whose Superficies is equal to the Superficies of an *Icosaedron* given.

$$\begin{array}{lcl}
 & \left. \begin{array}{l} 1 \\ 2 \end{array} \right| & \left. \begin{array}{l} \frac{15sy^2}{S} \\ Sw^2\sqrt{3} \end{array} \right\} = \text{Superf. of the } \left\{ \begin{array}{l} \text{Dodecaed. } 166 \\ \text{Icosaed. } 183 \end{array} \right. \\
 1 = 2 & 3 & \frac{15sy^2}{S} = Sw^2\sqrt{3}, \text{ per Hyp.} \\
 15sy^2 & 4 & 15sy^2 = 5Sw^2\sqrt{3}. \\
 4 \div 15s & 5 & y^2 = \frac{Sw^2\sqrt{3}}{3s}. \\
 5w^2 & 6 & y = \sqrt{\frac{Sw^2\sqrt{3}}{3s}}.
 \end{array}$$

T H E O R E M.

Divide a fifth Part of the Superficies of the *Icosaedron*, by three times the Sine of 54 Degrees, and the Square Root of that Quotient will be the side of the *Dodecaedron*.

H h 2

P R O B.

P R O B. XXII.

To find the Diameter of a *Sphere*, whose Solidity is a mean Proportional between its circumscribing *Cylinder*, and inscribed *Tetraedron*.

	1	$2 R^3 e : \frac{4 R^3 e}{3} :: \frac{4 R^3 e}{3} : \frac{a^3}{6\sqrt{2}}$ (per Hypoth.)
1 ∴	2	$\frac{2 R^3 a^3 e}{6\sqrt{2}} = \frac{16 R^6 e^2}{9}$
2 Red.	3	$18 R^3 a^3 e = 96 R^6 e^2 \sqrt{2}$
$3 \div 6 R^3 e$	4	$3 a^3 = 16 R^3 e \sqrt{2}$
$4 \div 2 e \sqrt{2}$	5	$\frac{3 a^3}{2 e \sqrt{2}} = 8 R^3$
$5 uv 3$	6	$2 R = a \sqrt[3]{\frac{3}{2 e \sqrt{2}}}$

T H E O R E M.

Divide 3, by twice the Rectangle of the Ratio of the Diameter of a Circle to the Circumference, into the Square Root of 2 ; multiply the Cube Root of that Quotient into the side of the *Tetraedron*, and the Product will be the Diameter required.

P R O B. XXIII.

To find the Diameter of a *Sphere*, whose Superficies is equal to the Superficies of a *Tetraedron*,
and

and Cube added together, both being inscribed in the same *Sphere*.

For the Diameter sought put $2r$.

$$\begin{array}{lcl}
 & \left. \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} \frac{8R^2\sqrt{3}}{3} \\ 8R^2 \end{array} & \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} = \text{Superf. of the } \left\{ \begin{array}{l} \text{Tetraed. } 115 \\ \text{Cube. } 127 \end{array} \right. \\
 1 + 2 & 3 & \frac{8R^2\sqrt{3}}{3} + 8R^2 = 4r^2e. \\
 3 \text{ Red.} & 4 & 4R^2 \times \frac{6+2\sqrt{3}}{3e} = 4r^2. \\
 4 \text{ w } 2 & 5 & 2R \times \sqrt{\frac{6+2\sqrt{3}}{3e}} = 2r.
 \end{array}$$

THEOREM.

Divide the Sum of 6, and twice the Square Root of 3, by three times the Ratio of the Diameter of a Circle to its Circumference; then multiply the Square Root of that Quotient into the Diameter of the circumscribing *Sphere*, and that Product will be the Diameter required.

PROB. XXIV.

To find the Side of a *Tetraedron*, whose Superficies is a mean Proportional between the Superficies of its circumscribing *Sphere*, and the Superficies of another *Sphere* whose Diameter is $2r$.

For

For the Diameter of the circumscribing *Sphere*,
put $2R$.

	1	$4R^2e : \frac{8R^2\sqrt{3}}{3} :: \frac{8R^2\sqrt{3}}{3} : 4r^2e.$
		(per Hyp.
1 st	2	$16R^2r^2e^2 = \frac{8R^2\sqrt{3}}{3} \times \frac{8R^2\sqrt{3}}{3}.$
2 nd	3	$4Rre = \frac{8R^2\sqrt{3}}{3}.$
But	4	$8R^2 = 3a^2. \quad 108$
4, 3	5	$4Rre = a^2\sqrt{3}.$
5 th & 6 th &c.	6	$a = 2\sqrt{\frac{Rre}{\sqrt{3}}}.$

T H E O R E M.

Multiply the Radius of the circumscribing *Sphere*, into the Circumference of the greatest Circle of the other *Sphere*, and divide that Product by the Square Root of 3; then twice the Square Root of that Quotient will be the side of the *Tetraedron*.

P R O B. XXV.

To cut a Cone NPO , by a Plane QDB parallel to the Base, so that the Superficies of the Plane, may be a mean Proportional between the Superficies's of the two Parts of the Cone.

For

Algebraically Demonstrated.

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For the Diameter of the Cone's Base put $2R$, and for the side NP put s , and for QP , the Distance of the Plane from the Vertex, put x .

	1	$R s e =$ Superficies of the Cone (NPO 45.
	2	$s : R :: x : \frac{R x}{s} = QD$, per (4. 6.
	3	$\frac{R x^2 e}{s} =$ Superficies of the Cone (QPB . 45.
1 — 3	4	$\frac{R s^2 e - R x^2 e}{s} =$ Superficies of (the Cone $NQBO$.
2 $\odot$ 2 $\& x e$	5	$\frac{R^2 x^2 e}{s^2} =$ Superf. of the Plane (BDQ . 31.
3. 5. 4.	6	$\frac{R x^2 e}{s} : \frac{R^2 x^2 e}{s^2} :: \frac{R^2 x^2 e}{s^2} :$ $\left(\frac{R s^2 e - R x^2 e}{s} \right.$ per <i>Hyp.</i>
6 $\therefore$	7	$\frac{R^2 s^2 x^2 e^2 - R^2 x^4 e^2}{s^2} = \frac{R^4 x^4 e^2}{s^4}$
7 Red.	8	$s^2 - x^2 = \frac{R^2 x^2}{s^2}.$
8 $\times s^2$	9	$s^4 - s^2 x^2 = R^2 x^2.$
9 $\div s^2 x^2$	10	$R^2 x^2 \div s^2 x^2 = s^4.$
10 $\div \& c.$	11	$x = \frac{s^2}{\sqrt{R^2 + s^2}}.$

THEO.

T H E O R E M.

To the Square of the Radius of the Cone's Base, add the Square of the flant Height; by the Square Root of that Sum divide the Square of the said Height, and the Quotient is the Distance of the cutting Plane from the Vertex.

P R O B. XXVI.

Fig. 18. Let there be a regular Polygon, whose Sides are measured by the Number Four, and let it be turned round about the Diameter AE , and suppose the Superficies of the Body generated by that Rotation, to be equal to the Superficies of a Circle, whose Radius is R ; what is the Diameter AE of the circumscribing Circle, and AB , the Side of the Polygon?

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Algebraically Demonstrated.

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For AE put x , for the Sine of the Angle AEB , (which is suppos'd known) put s , and for the Sine of 90 Deg. put r .

	1	$BE = \frac{R^2}{x} \cdot 327$
	2	$r : x :: s : \frac{s x}{r} = AB$, per Trigo- (nometry.
2 . .	3	$\sqrt{x^2 - \frac{R^4}{x^2}} = AB$.
2 = 3	4	$\sqrt{x^2 - \frac{R^4}{x^2}} = \frac{s x}{r}$.
4 $\odot$ 2	5	$x^2 - \frac{R^4}{x^2} = \frac{s^2 x^2}{r^2}$.
5 Red,	6	$x^4 - R^4 = s^2 x^4$, because $r^2 =$ (1,00000, &c.
6 $\div$ —	7	$x^4 - s^2 x^4 = R^4$.
7 $\div$ 1 — S^2	8	$x^4 = \frac{R^4}{1 - s^2}$.
8 $\propto$ 4	9	$x = R \sqrt[4]{\frac{1}{1 - s^2}}$.

THEOREM.

From an Unit subtract the Square of the Natural Sine of the Angle AEB , by the Remainder divide an Unit ; then the Square Root of that Remainder multiplied into the Radius of the proposed Circle, will give AE , the Diameter of the Circle circumscribing the *Polygon*.

I

SCHO-

SCHOLIUM.

If we suppose the Angle AEB to be infinitely small, the Square of s will be nothing, and the foregoing Expression will be $x = R\sqrt{\frac{1}{1-0}}$, and $x=R$, i. e. the Body generated by the Rotation of that *Polygon* round its Axis AE , will be a Sphere, whose Superficies will be $x^2 e$, per *Art.* 103. exactly equal to the Superficies of a Circle, whose Radius is R , per *Art.* 31.

P R O B. XXVII.

Fig. 21. To cut a given Sphere $ACDEF$, by a Plane $Q N$, perpendicular to the Diameter AE , so that the Superficies of the Portion $ADFA$, may be to the Superficies of the Portion DEF , in the Ratio of x to Z .

Put $AO=y$, then is $EO = 2R-y$.

	1	$2R:AD::AD:y$	
	2	$2R:ED::ED:2R-y$	} (per <i>Cor.</i> 8. 6.
But {	3	$\overline{AD}^2 = 2Ry$	} is as the
	4	$\overline{ED}^2 = 4R^2 - 2Ry$	
		Portion $\left\{ \begin{array}{l} ADFA \\ DEF \end{array} \right\}$	382
	5	$2Ry:4R^2-2Ry::x:Z$, per	(Hypoth.
	6	$2Ry:4R^2::x:x+Z$	
6 ∴	7	$2Rxy + 2RZy = 4R^2x$	
7 ÷	8	$y = \frac{4R^2x}{2Rx + Z} = 2R \times \frac{x}{x+Z}$	

THE O-

THEOREM.

Divide one of the Terms of the given Ratio, by the Sum of the said Terms, multiply the Quotient into the Diameter of the Sphere, that Product will be the Distance of the cutting Plane from the Point *A*.

N. B. If you make the greater Term of the Ratio your Dividend, the *Theorem* will give the greater Segment of the Diameter ; and the contrary.

SCHOLIUM.

If the Ratio be a Ratio of Equality, *i. e.* if $x = Z$, then $y = 2 R \times \frac{x}{x + Z}$ is $= 2 R \times \frac{1}{2} = R$, *i. e.* the cutting Plane will pass through the Center of the Sphere.

Again ; If $Z = 0$, then $y = 2 R \times \frac{x}{x + Z}$ ($= 2 R \times \frac{x}{x + 0}$) is $= 2 R$, *i. e.* the cutting Plane must be a Tangent to the Sphere at *E*.

FINIS.



